



KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS



KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 11

2025

**Mathematics (L2FM)
Unit 24 Tasks – Part 1**

2026

DO NOT WRITE INSIDE



KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS



KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 11

2025

**Mathematics (L2FM)
Unit 24 Tasks – Part 2**

2026

DO NOT WRITE INSIDE



KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS



KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 11

2025

**Mathematics (L2FM)
Unit 24 Tasks – Part 3**

2026

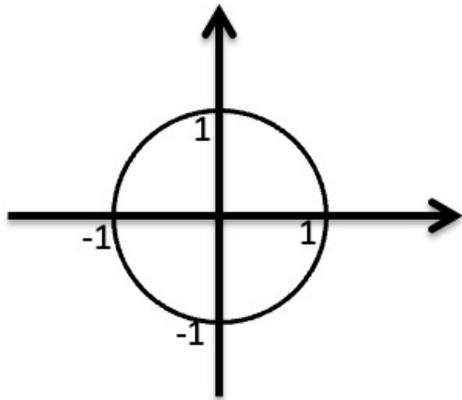
DO NOT WRITE INSIDE

Contents

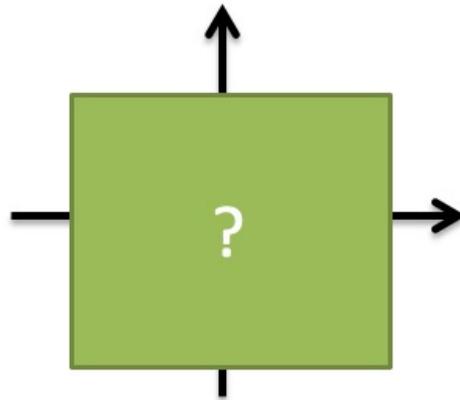
- 1 [Equations of Circles and Tangents](#)
- 2 [Advanced Equations of Circles \(L2FM Only\)](#)
- 3 [Advanced Simultaneous Equations](#)
- 4 [Advanced Sequences](#)
- 5 [Limiting Values of Sequences \(L2FM Only\)](#)
- 6 [Algebraic Proof](#)
- 7 [Advanced Vectors](#)

1 Equations of Circles and Tangents

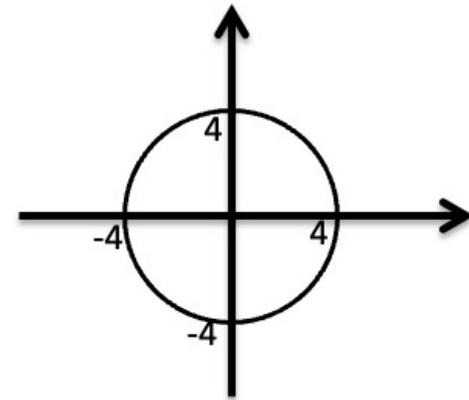
Fluency Practice



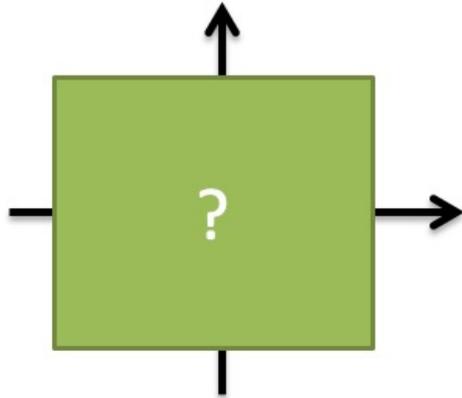
?



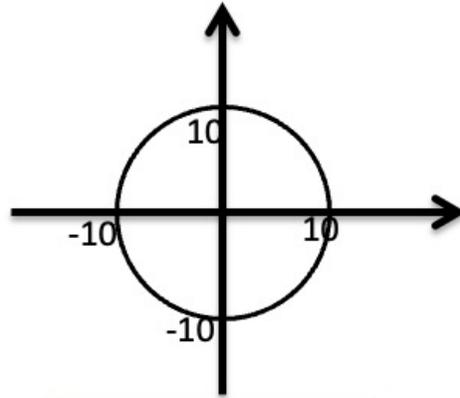
$$x^2 + y^2 = 9$$



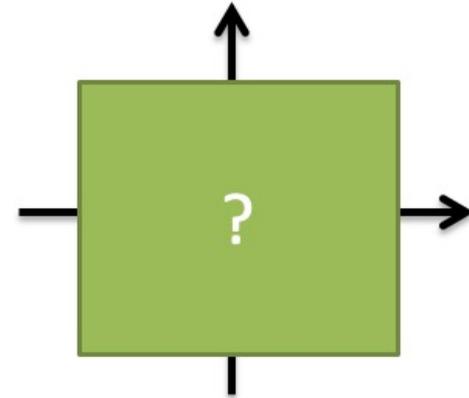
?



$$x^2 + y^2 = 64$$



?



$$x^2 + y^2 = 36$$

Fluency Practice

Find the radius of the circle: $x^2 + y^2 = 36$	Find the radius of the circle: $x^2 + y^2 = 45$
Find the diameter of the circle: $x^2 + y^2 = 100$	Find the area of the circle: $x^2 + y^2 = 144$
Find the circumference of this circle: $x^2 + y^2 = 9$	Find the area of the circle: $x^2 + y^2 = 48$
Write down the equation of the circle with centre $(0, 0)$ and diameter 800.	Write down the equation of the circle with centre $(0, 0)$ and area 201.
Write the equation of a circle with centre $(0, 0)$ and radius $\sqrt{11}$.	Write the equation of a circle with centre $(0, 0)$ and radius $2\sqrt{3}$.
The point $(3, a)$ is on the circle $x^2 + y^2 = 25$ Find the possible values of a .	The point $(b, 12)$ is on the circle $x^2 + y^2 = 169$ Find the possible values of b .

Fluency Practice

Plotting Different Graphs

①

On the grid, plot the graph

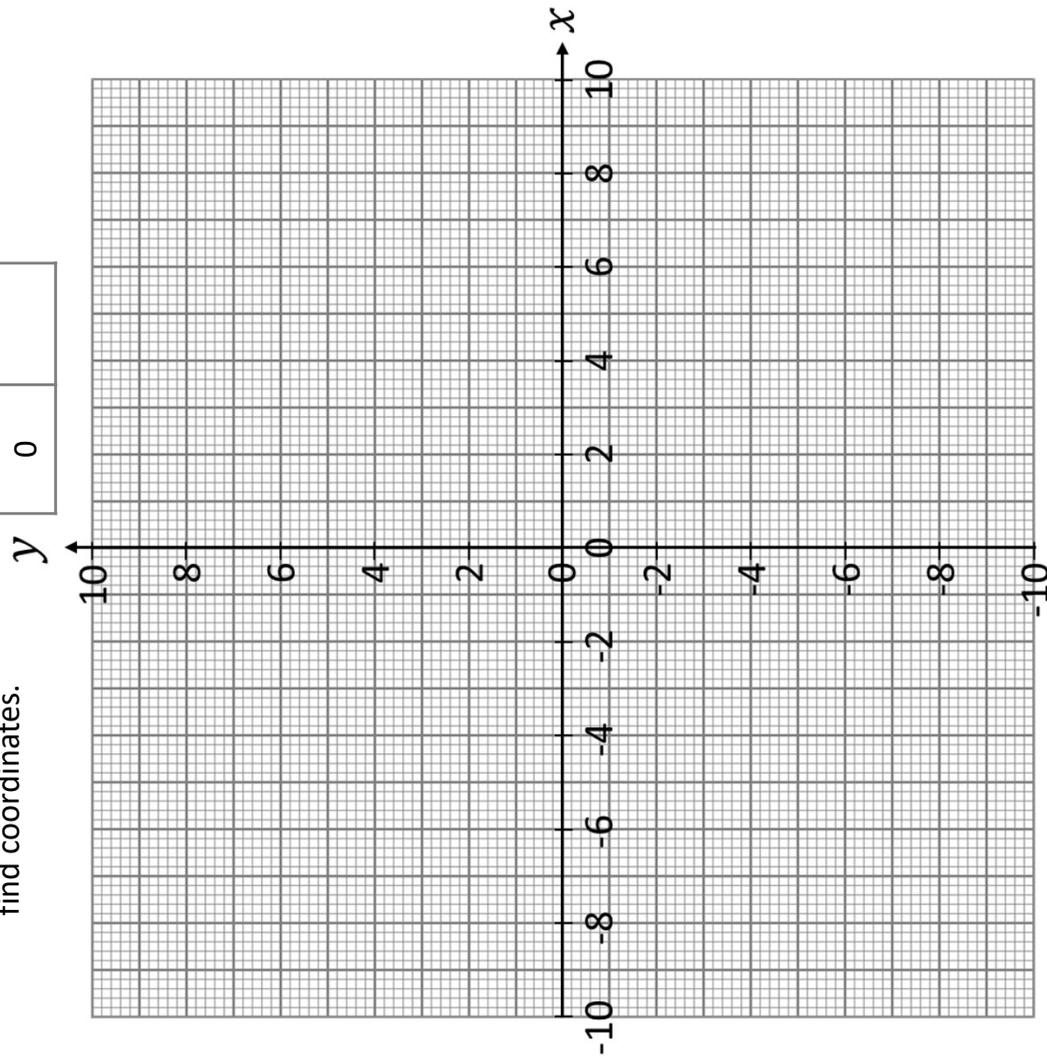
$$x^2 + y^2 = 100$$

If $x = 0$,
what values will y be?

Form equations to help
find coordinates.

x	y
-10	
-8	
-6	
-4	
-2	
0	

x	y
10	
8	
6	
4	
2	



Fluency Practice

②

Plotting Different Graphs

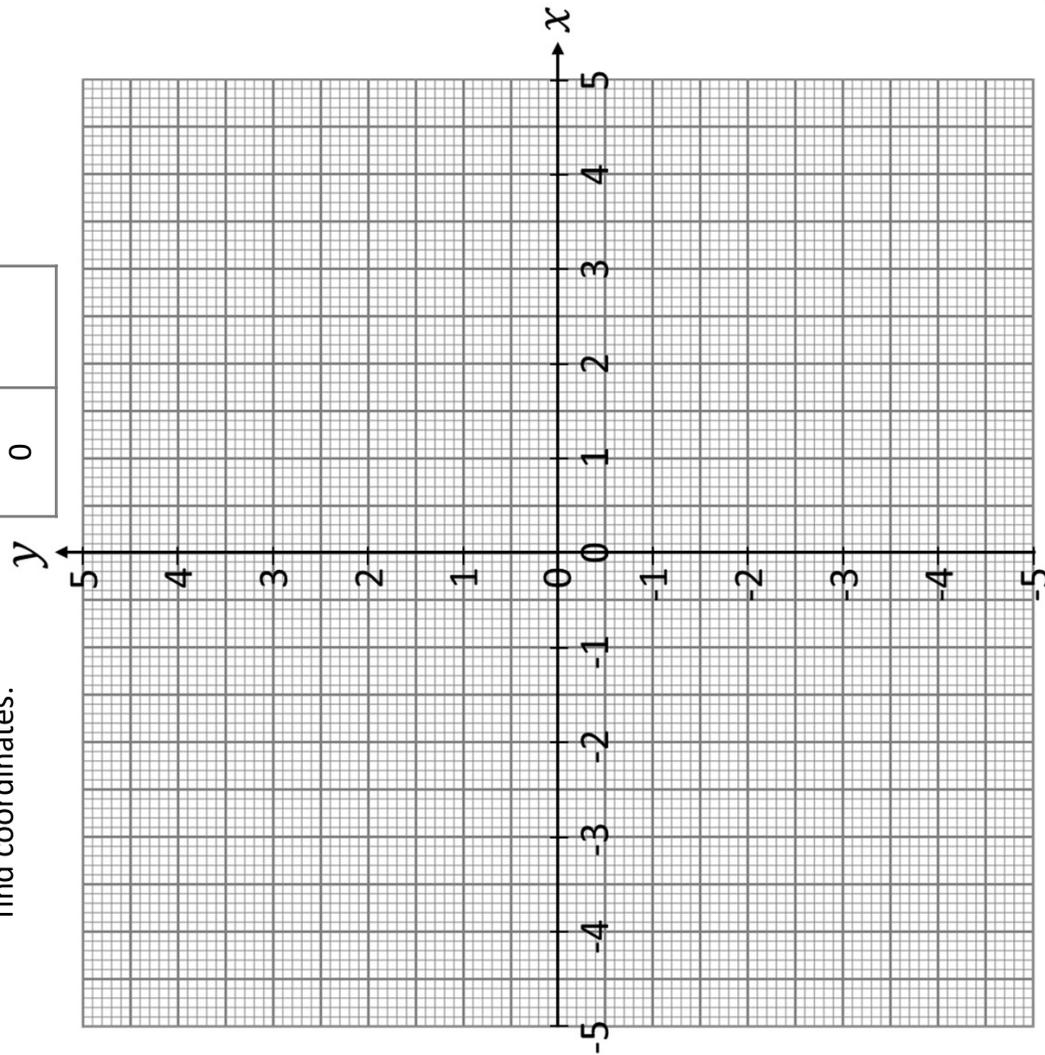
On the grid, plot the graph

$$x^2 + y^2 = 16$$

If $x = 0$,
what values will y be?

Form equations to help
find coordinates.

x	y
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	



Fluency Practice

Plotting Different Graphs

③ 

On the grid, plot the graph

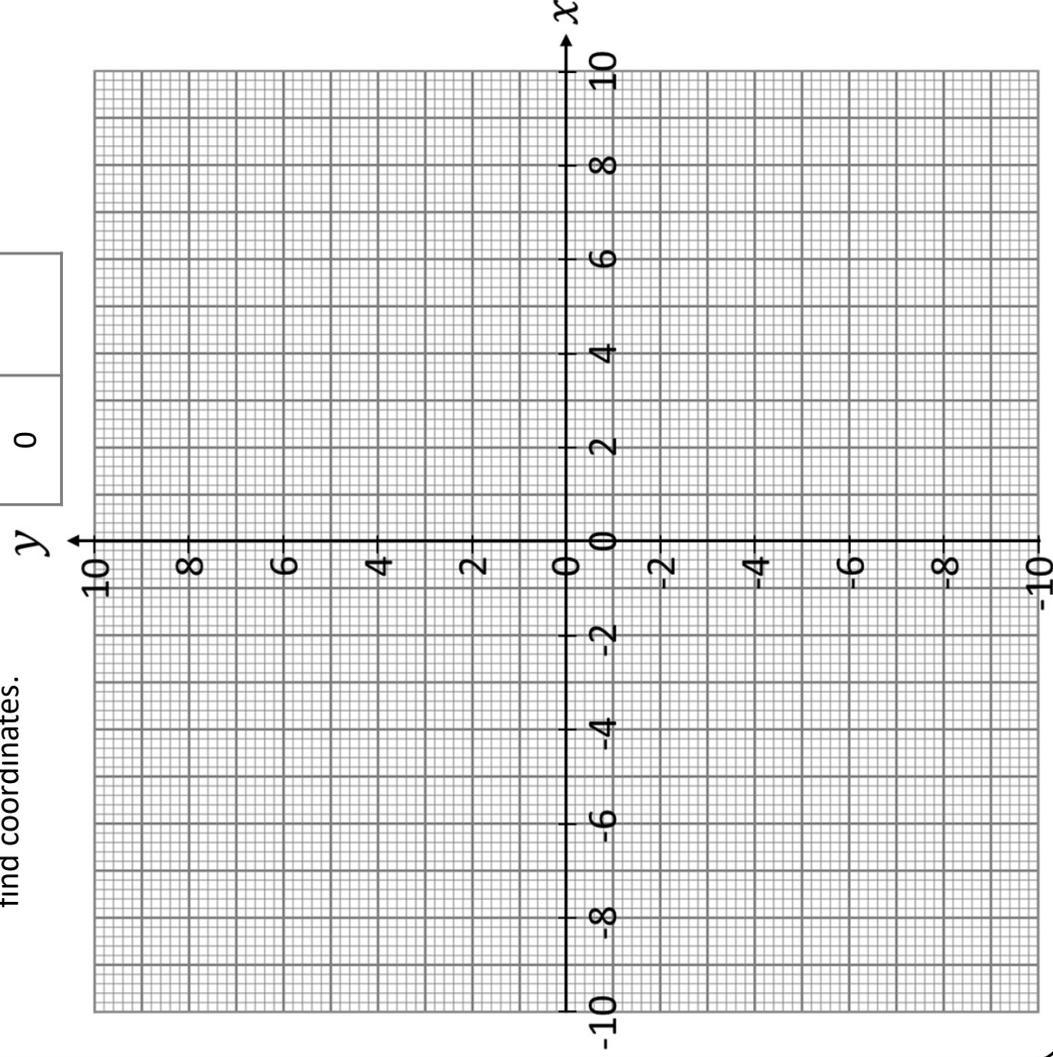
$$x^2 + y^2 = 64$$

If $x = 0$,
what values will y be?

Form equations to help
find coordinates.

x	y
-8	
-6	
-4	
-2	
0	

x	y
8	
6	
4	
2	



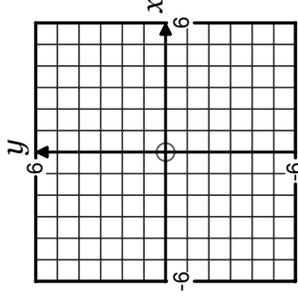
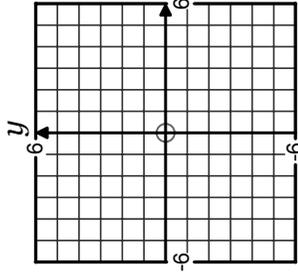
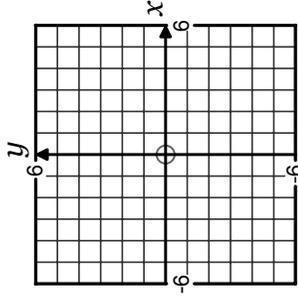
Purposeful Practice

1. Draw the graph of each function:

a) $x^2 + y^2 = 4$

b) $x^2 + y^2 = 25$

c) $x^2 + y^2 = 4^2$



2. A circle has equation $x^2 + y^2 = 64$. Work out the diameter of this circle.

3. Write down the equation of a circle centred on the origin with:

a) a radius of 7

b) a diameter of 20

c) an area of 36π

4. Which of the following points lie on the circle with equation $x^2 + y^2 = 100$?
Select all that apply.

a) (10, 1)

b) (-8, 6)

c) (7, -7)

d) (5, $5\sqrt{3}$)

5. Which of the following points lie on the circle with equation $x^2 + y^2 = 80$?
Select all that apply.

a) (0, $4\sqrt{5}$)

b) ($3\sqrt{10}$, $-\sqrt{10}$)

c) (-7, $-\sqrt{31}$)

6. Ravi draws the graphs with equations $x^2 + y^2 = 169$ and $y = -5$.
Work out the coordinates of the points where Ravi's two graphs intersect.

7. Circle **C** has equation $x^2 + y^2 = 81$.

Write down the equations of the two horizontal lines that are tangent to **C**.

8. Which of the following are equations of circles? Select all that apply.

a) $x^2 + y = 121$

b) $(x + y)^2 = 9$

c) $x^2 + y^2 - 24 = 0$

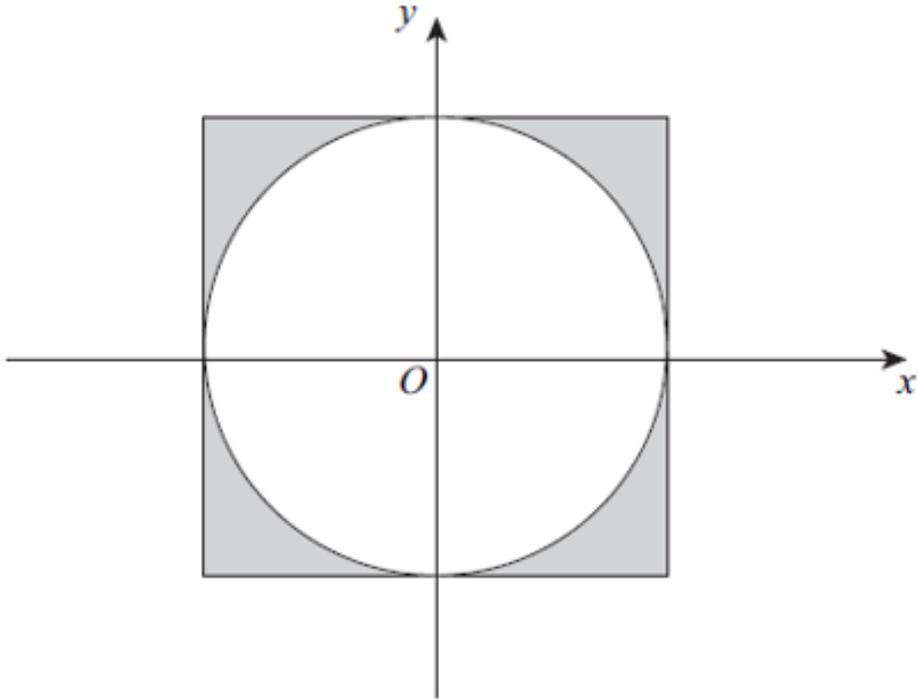
d) $x = 16 - y^2$

e) $20 = y^2 + x^2$

f) $x^2 + y^2 = \sqrt{5}$

Fluency Practice

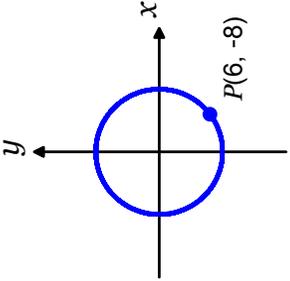
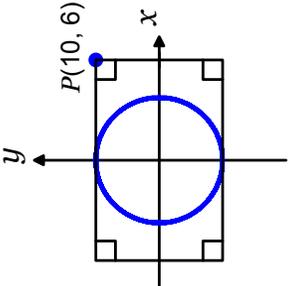
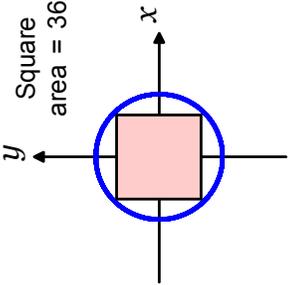
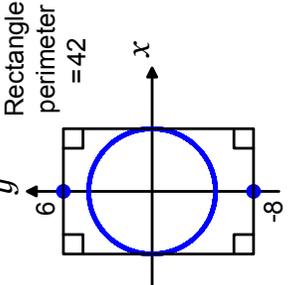
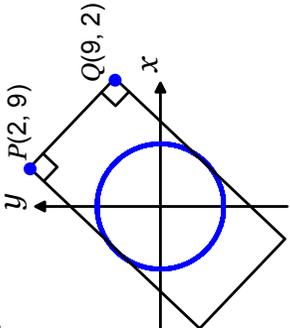
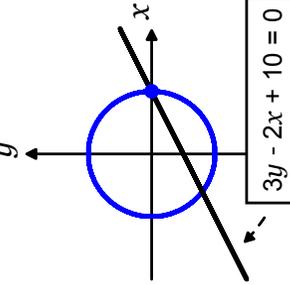
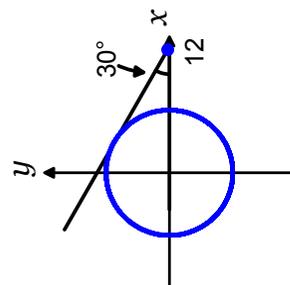
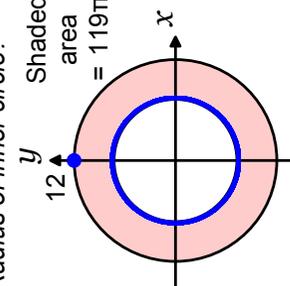
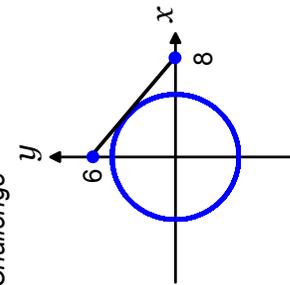
- 1) Write down the centre and radius of the circle with equation $x^2 + y^2 = 36$
- 2) A circle has the equation $x^2 + y^2 = 36$. Work out its circumference.
- 3) The circle $x^2 + y^2 = 25$ touches each side of the square as shown. Work out the total shaded area.



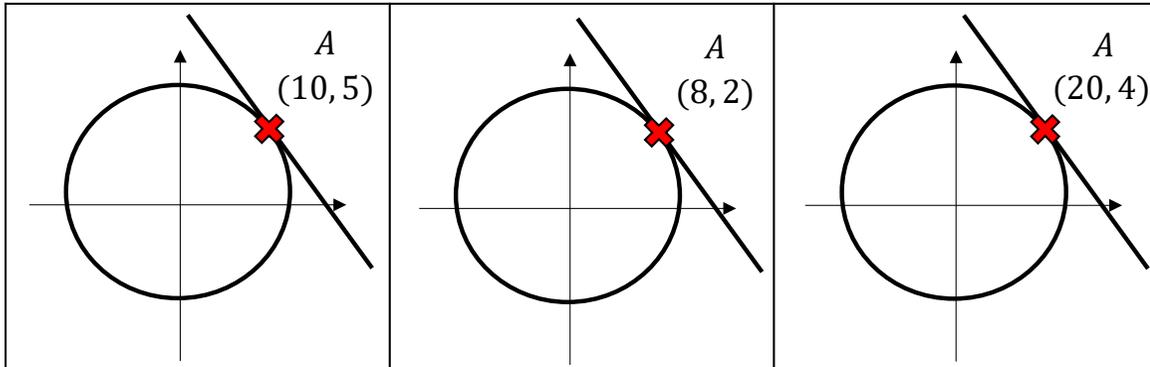
Purposeful Practice

Circle graphs: find the radius

Work out the radius of each circle graph centred on the origin. Lines that look like tangents are tangents.

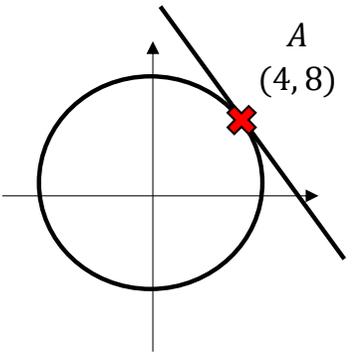
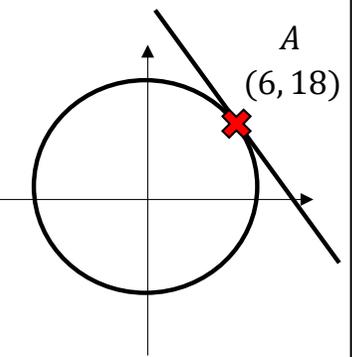
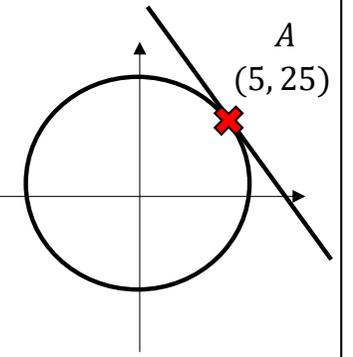
<p>A</p> 	<p>B</p> 	<p>C</p>  <p>Square area = 36</p>
<p>D</p>  <p>Rectangle perimeter = 42</p>	<p>E</p> 	<p>F</p>  <p>$3y - 2x + 10 = 0$</p>
<p>G</p> 	<p>H Radius of inner circle?</p>  <p>Shaded area = 119π</p>	<p>I Challenge</p> 

Fluency Practice

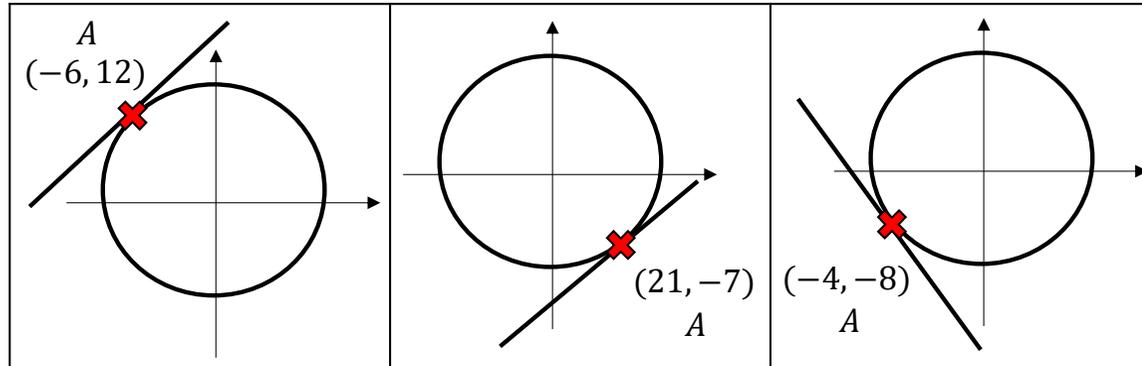


What is the gradient between A and the origin?			
What is the gradient of the tangent?			
What is the equation of the tangent (so far)?			
Substitute the co-ordinate to find the value of c			
Final Answer: The equation of the tangent is...			

Fluency Practice

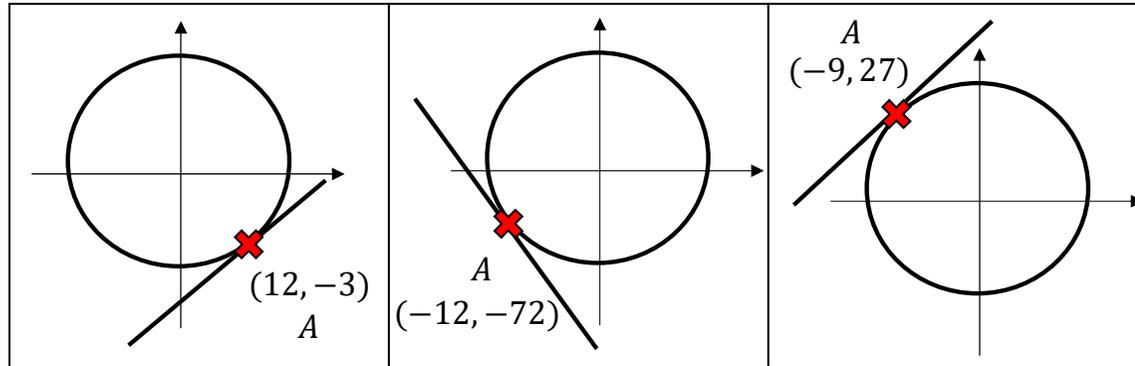
			
What is the gradient between A and the origin?			
What is the gradient of the tangent?			
What is the equation of the tangent (so far)?			
Substitute the co-ordinate to find the value of c			
Final Answer: The equation of the tangent is...			

Fluency Practice



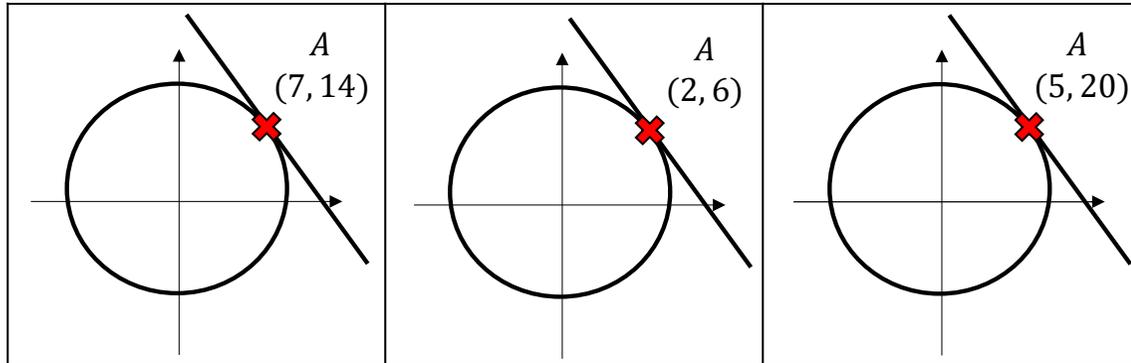
What is the gradient between A and the origin?			
What is the gradient of the tangent?			
What is the equation of the tangent (so far)?			
Substitute the co-ordinate to find the value of c			
Final Answer: The equation of the tangent is...			

Fluency Practice



What is the gradient between A and the origin?			
What is the gradient of the tangent?			
What is the equation of the tangent (so far)?			
Substitute the co-ordinate to find the value of c			
Final Answer: The equation of the tangent is...			

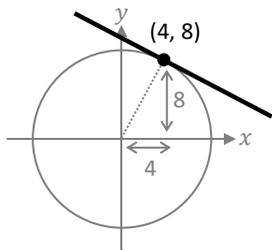
Fluency Practice



What is the gradient between A and the origin?			
What is the gradient of the tangent?			
What is the equation of the tangent (so far)?			
Substitute the co-ordinate to find the value of c			
Final Answer: The equation of the tangent is...			

Fluency Practice

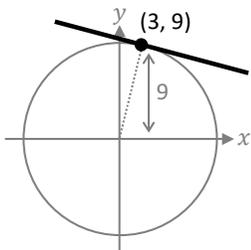
Rad. Grad. = 2 Tan. Grad. = $-\frac{1}{2}$



$y = -\frac{1}{2}x + 8$

$y = -\frac{1}{2}x + 10$

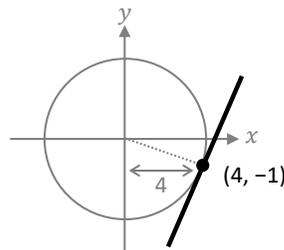
Rad. Grad. = Tan. Grad. =



$y = -\frac{1}{3}x + 10$

$y = -\frac{1}{3}x + 11$

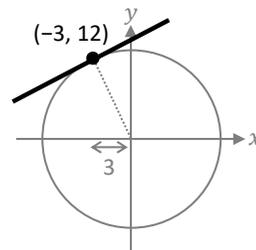
Rad. Grad. = $-\frac{1}{4}$ Tan. Grad. =



$y = 4x - 22$

$y = 4x - 17$

Rad. Grad. = -4 Tan. Grad. =

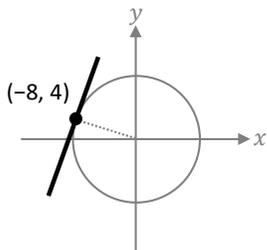


$y = \frac{1}{4}x + 13$

$y = \frac{1}{4}x + 12\frac{3}{4}$

**Tangent
Match**

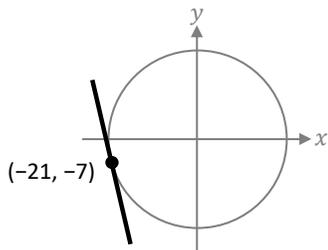
Rad. Grad. = Tan. Grad. =



$y = x + 20$

$y = x + 24$

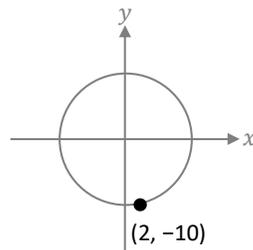
Rad. Grad. = Tan. Grad. =



$y = x - 56$

$y = x - 54$

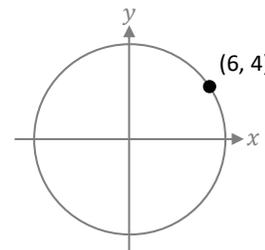
Rad. Grad. = Tan. Grad. =



$y = x - 10.2$

$y = x - 10.4$

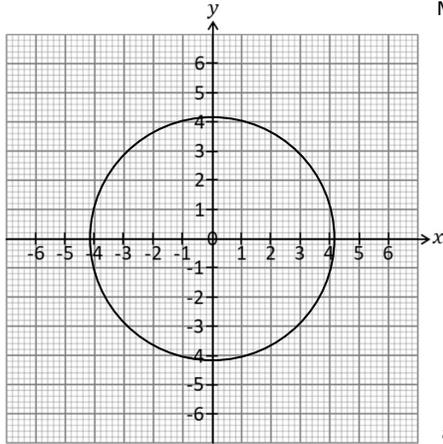
Rad. Grad. = Tan. Grad. =



Purposeful Practice

①

The graph $x^2 + y^2 = 18$ is on the grid.



Mark point **A** on the circle at (3, 3), then draw line **OA** from the origin (**O**) to point **A**.

What is the equation of line **OA**?

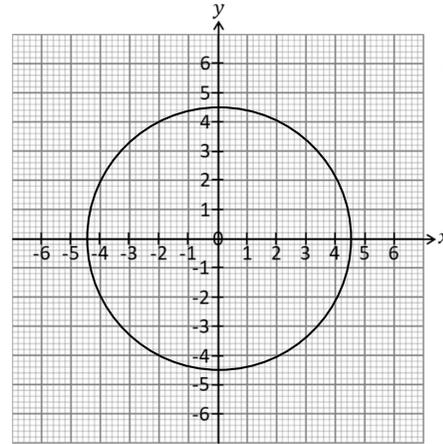
Plot the line $y = 6 - x$

How are the gradients of these graphs related?

$y = 6 - x$ is _____ to the circle.

Lines around Circles

The graph $x^2 + y^2 = 20$ is on the grid.



Mark point **B** on the circle at (2, 4), & draw line **OB** from the origin.

What is the gradient of the line **OB**?

What is the gradient of a line perpendicular to **OB**?

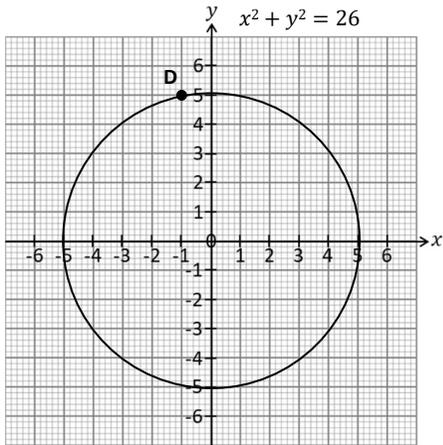
Plot & label line **CD** - the line that is perpendicular to **OB** & passes through **B**.

Draw & label the line that is parallel to **CD** & touches the circle at one point.

Draw & label the line that is parallel to $y = 6 - x$ & touches the circle at one point.

Where on the circle will a tangent have a gradient of 2? _____

③



Sketch the line **OD**.

Find the gradient of this radius, is it positive or negative?

The tangent to the circle at **D** has a gradient of $\frac{1}{5}$ and passes through (-1, 5).

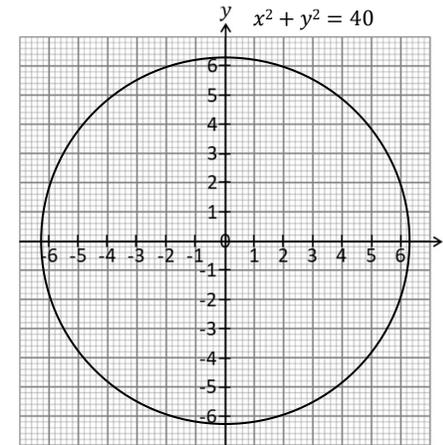
We can express this as:

$$y = \frac{1}{5}x + c$$

Plot the tangent & calculate its y-intercept.

②

④



Find the equation of the tangent at (6, -2)

Find the equation of the tangent at (-2, 6)

Purposeful Practice

(a) The point P $(6, -2)$ lies on the circle with equation $x^2 + y^2 = 40$. Find the gradient of the tangent to the circle at point P.

(b) The point Q $(2, 1)$ lies on the circle with equation $x^2 + y^2 = 5$. Find the gradient of the tangent to the circle at point Q.

(a) The point A $(2, -5)$ lies on the circle with equation $x^2 + y^2 = 29$. Find the equation of the tangent to the circle at point A.

(b) The point B $(-3, -2)$ lies on the circle with equation $x^2 + y^2 = 13$. Find the equation of the tangent to the circle at point A.

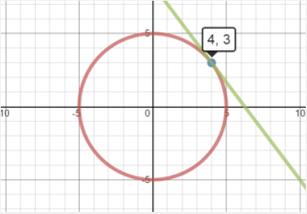
(a) The point R $(-2, -3)$ lies on the circle with centre $(1, 2)$ and radius $\sqrt{34}$. Find the equation of the tangent to the circle at point R.

(b) The point S $(7, 1)$ lies on the circle with centre $(x - 3)^2 + (y + 2)^2 = 25$. Find the equation of the tangent to the circle at point S.

(a) Point P $(-2, 8)$ lies on a circle with centre $(-1, 6)$. Point Q with coordinates $(a, 5)$ lies on the tangent to the circle at P. Find the value of a .

(b) Point A $(-1, -3)$ lies on the circle with equation $(x - 3)^2 + (y + 2)^2 = 17$. The line L is the tangent to the circle at point A. Find the coordinates of the point where line L crosses the x -axis.

Fluency Practice

Question	Radius	Sketch	Gradient of radius at point	Gradient of tangent at point	Equation of tangent at point
Example: Find the equation of the tangent to $x^2 + y^2 = 25$ at $(4, 3)$	5		$\frac{3}{4}$	$-\frac{4}{3}$	$y = -\frac{4}{3}x + \frac{25}{3}$
1. Find the equation of the tangent to $x^2 + y^2 = 5$ at $(2, 1)$					
2. Find the equation of the tangent to $x^2 + y^2 = 100$ at the point on the circumference with x -coordinate 6 and a positive y -coordinate					
3. Find the equation of the tangent to $x^2 + y^2 = 45$ at $(-6, 3)$					

Fluency Practice

4. Find the equation of the tangent to
 $x^2 + y^2 - 20 = 0$ at $(-4, -2)$

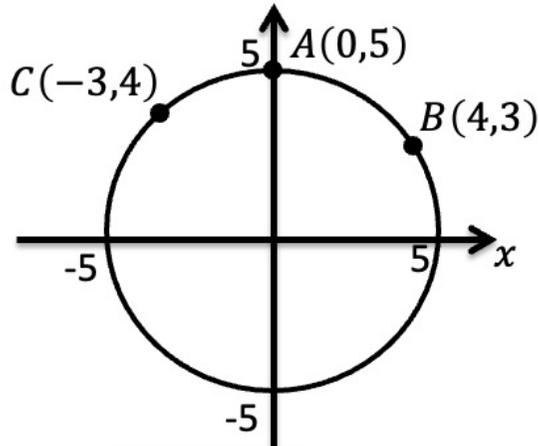
5. Find the equation of the tangent to
 $x^2 + y^2 = 13$ at the point on the
circumference with x -coordinate 3
and a negative y -coordinate

6. Find the equation of the tangent to
the circle with centre $(0, 0)$ and
diameter $\sqrt{32}$ at the point $(2, 2)$

7. Find the equation of the tangent to
 $x^2 + y^2 = 25$ at the point $(5, 0)$

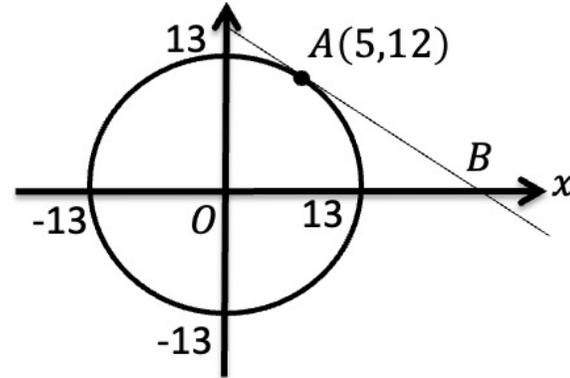
Purposeful Practice

- 1 The diagram shows the circle with equation $x^2 + y^2 = 25$. What is the equation of the tangent at the following points?



- a $A(0,5)$:
- b $B(4,3)$:
- c $C(-3,4)$:

2



The tangent to the above circle at the point $A(5,12)$ intersects the x axis at the point B .

- a Find the equation of the tangent to the circle at the point A .

- b) Find the area of triangle OAB .

b



The line l is tangent at the point $P(x, y)$ to the circle with equation $x^2 + y^2 = 1$. The gradient of l is $-\frac{1}{2}$. Determine the point $P(x, y)$.

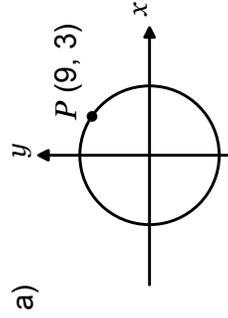
Purposeful Practice

WARM - UP

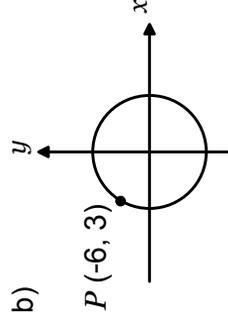
- Work out the gradient of the line from the origin to each of these points:
 a) (5, 8) b) (-3, 7) c) $(-2, \sqrt{3})$ d) $(-2\sqrt{2}, 4)$
- In each case, the point P lies on the circle C. Work out a , b and c .
 a) C: $x^2 + y^2 = 100$ b) C: $x^2 + y^2 = 4^2$ c) C: $x^2 + y^2 - 25 = 0$
 P: (-6, a) P: (2, b) P: (-2, c)
 $a > 0$ $b > 0$ $c < 0$
- Work out the gradient of a line **perpendicular** to a line with gradient:
 a) -4 b) $\frac{2}{5}$ c) $-\frac{3}{4}$ d) $\frac{1}{5}$
- In each case, the point P lies on a line with gradient m . Work out the equation of the line in the form $y = mx + c$.
 a) P(2, 5), $m = 3$ b) P(-3, 4), $m = -1$ c) P(-3, -2), $m = \frac{1}{4}$

QUESTIONS

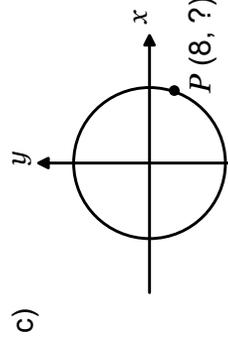
- Work out the equation of the tangents to these circles at the given point P. Give your answers in the form $y = mx + c$.



$$x^2 + y^2 = 90$$



$$x^2 + y^2 = 45$$



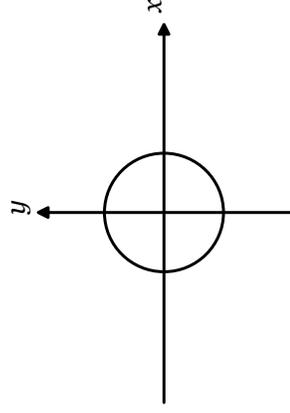
$$x^2 + y^2 = 80$$

- Circle C has equation $x^2 + y^2 = 400$.
 Point P(a , b) lies on C such that $a = -12$ and $b < 0$.
 a) Work out the value of b .
 b) Work out the equation of the tangent to C at P in the form $y = mx + c$.

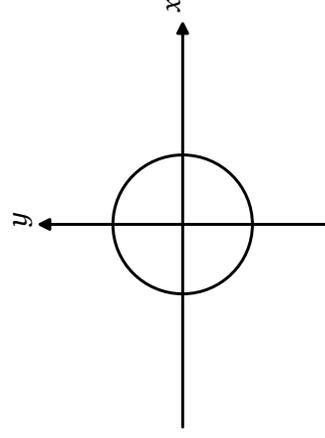
Purposeful Practice

3. Circle C has equation $x^2 + y^2 = 72$.
Two points on C have a tangent with a gradient of 0.

Work out the coordinates of these two points.

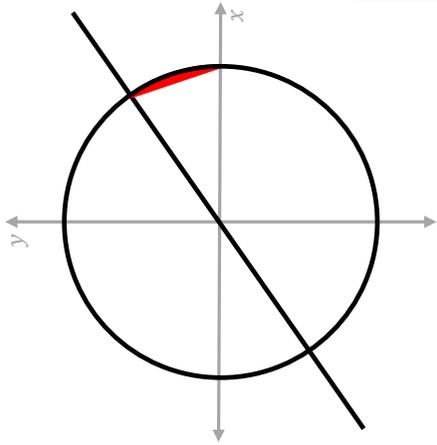


4. Circle C is centred on the origin and has a radius of 15 units.
Point P(α , b) lies on C, such that $\alpha = 12$ and $b > 0$.
Line T is the tangent to C at P.
- Work out the equation of T.
 - Work out the coordinates of the point at which T meets the y -axis.
 - Work out the coordinates of the point at which T meets the x -axis.
 - Work out the area of the triangle formed by the x -axis, the y -axis and T.
5. Circle C has equation $x^2 + y^2 = 32$.
Two different lines are both tangents to C and have a gradient of 1.
Work out the coordinates of the points where the tangents meet C.
6. A circle is centred on the origin.
The points P(-15, -20) and Q(-24, c) lie on the circle, where $c > 0$.
Work out the equation of the tangent to the circle at Q.
7. Circle C has equation $x^2 + y^2 = 50$.
O is the origin (0, 0).
P(5, s) is a point on the circle.
Q(1, t) is a point on the circle.
 $s > 0$ and $t < 0$.
- Work out the values of s and t .
 - Work out the equations of the tangents to C at P and Q.
 - Work out the coordinates of the point R, where the two tangents meet.
 - Given that $\hat{P}OQ = 127^\circ$, work out $\hat{P}RQ$.



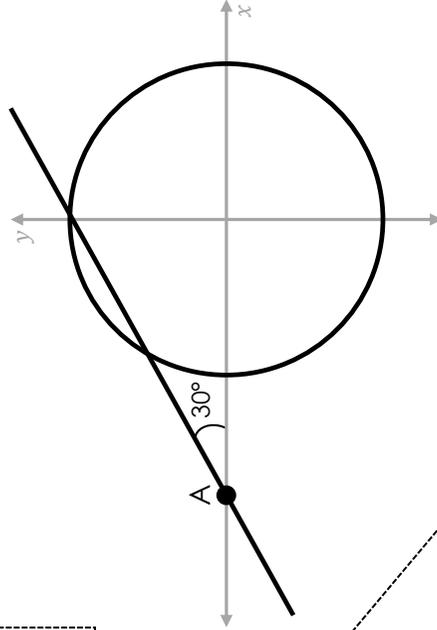
Purposeful Practice

8 fiendishly difficult circle questions for GCSE

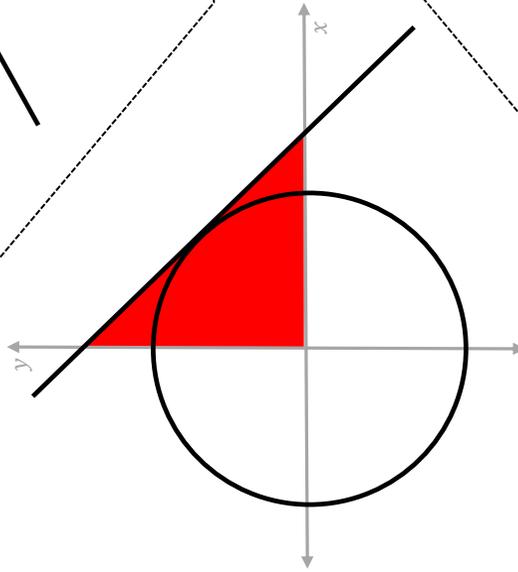


The point A has coordinates $(-6, 0)$. The straight line shown makes an angle of 30° with the x -axis. What is the equation of the circle?

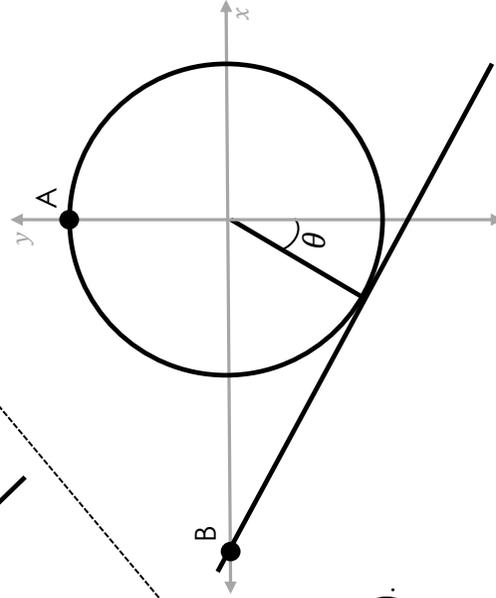
The equation of the circle is $x^2 + y^2 = 36$. The equation of the straight line is $y = \frac{\sqrt{3}}{3}x$. What is the area of the shaded segment?



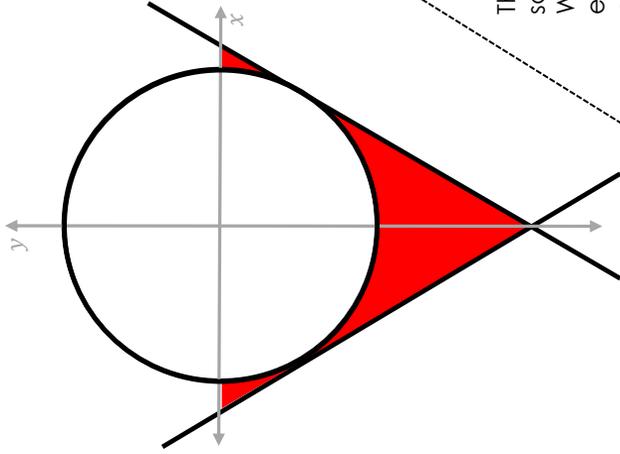
A tangent meets the circle at $(5, 12)$. What is the area of the shaded triangle?



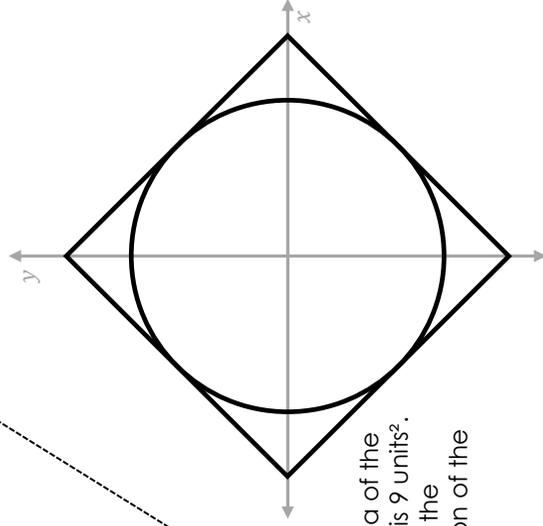
The point A has coordinates $(0, \frac{5}{3})$. The point B has coordinates $(-\frac{8}{3}, 0)$. Calculate the angle labelled θ .



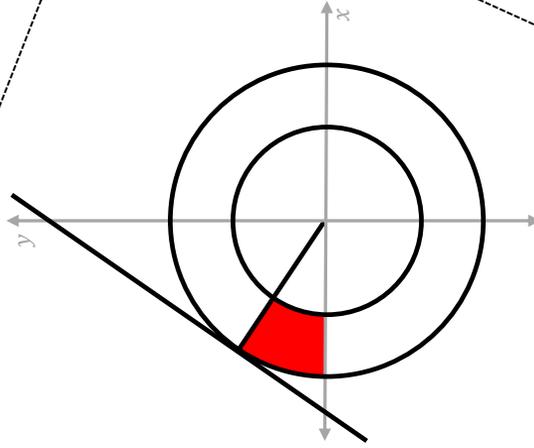
Purposeful Practice



One tangent meets the circle at $(6, -8)$, and another meets the circle at $(-6, -8)$. What is the total area of the shaded sections?



The area of the square is 9 units². What is the equation of the circle?



Two concentric circles are shown, one is an enlargement of the other by a scale factor 3 and has the equation $x^2 + y^2 = 144$.

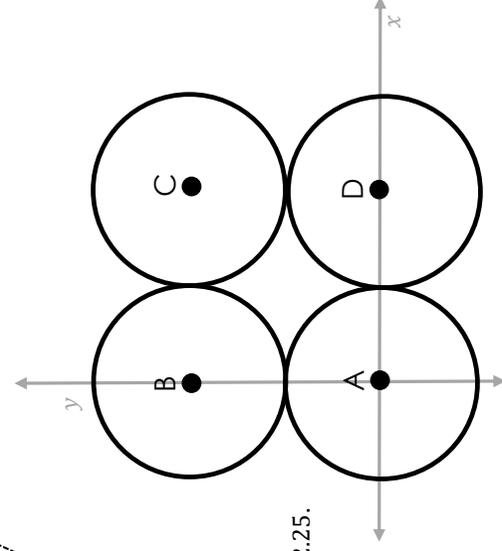
The tangent shown has the equation $y = \sqrt{3}x + 24$.

Calculate the area of the shaded section.

Four congruent circles are shown, with centres A, B, C, and D marked.

One circle has the equation $y^2 + x^2 = 2.25$.

Calculate the distance A to C



2 Advanced Equations of Circles (L2FM Only)

Fluency Practice

Find the centre and radius of each of these circles:

(a) $(x - 1)^2 + (y - 1)^2 = 9$

(b) $(x - 1)^2 + (y - 1)^2 = 25$

(c) $(x - 3)^2 + (y - 2)^2 = 25$

(d) $(x + 3)^2 + (y + 2)^2 = 16$

(e) $x^2 + (y + 2)^2 = 16$

(f) $(x - 4)^2 + y^2 = 36$

Write down the equation of the circle with:

(a) Centre (1, 2) and radius 8

(b) Centre (7, 3) and radius 2

(c) Centre (-2, 5) and radius 5

(d) Centre (-5, -1) and radius 4

(e) Centre (3, -6) and radius $\sqrt{7}$

(f) Centre (0, -4) and radius $\sqrt{20}$

Find the centre and radius of the circle with equation:

(a) $x^2 + y^2 - 2x + 8y - 8 = 0$

(b) $x^2 + y^2 + 12x - 4y = 9$

(c) $x^2 + y^2 - 22x - 6y + 40 = 0$

(d) $x^2 + y^2 - 4x - 11 = 0$

(a) Show that the point (2, 10) lies on the circle with equation

$$(x - 2)^2 + (y - 7)^2 = 9$$

(b) A circle has centre (5, 10). The point (2, 14) lies on the circumference of the circle. Find the equation of the circle.

Fluency Practice

Question 1: Write down the (i) centre and the (ii) radius for each of these circles.

(a) $(x - 3)^2 + (y - 2)^2 = 100$ (b) $(x - 1)^2 + (y + 3)^2 = 16$

(c) $(x + 5)^2 + (y - 7)^2 = 4$ (d) $(x + 6)^2 + (y + 8)^2 = 1$

(e) $x^2 + y^2 = 25$ (f) $(x + 4)^2 + y^2 = 9$

(g) $(x + 2)^2 + (y - 7)^2 = 5$ (h) $(x - 6)^2 + (y - 1)^2 = 20$

Question 2: Write the equations of each of these circles.

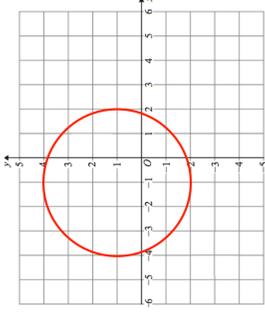
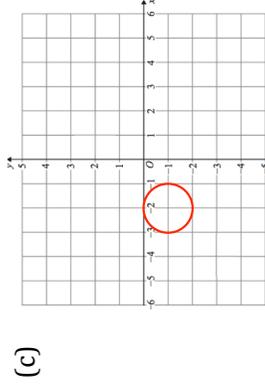
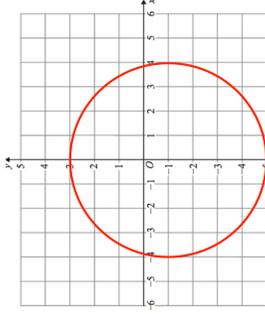
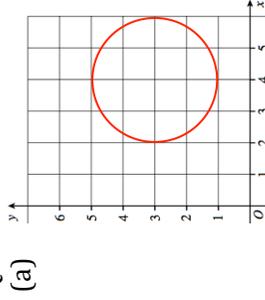
(a) Centre is (2, 7) and has a radius of 6. (b) Centre is (4, -2) and has a radius of 3.

(c) Centre is (-1, 3) and has a radius of 4. (d) Centre is (-1, -2) and has a radius of 9.

(e) Centre is (0, 5) and has a radius of 5. (f) Centre is the origin and has radius 8.

(g) Centre is (11, 0) and has a radius of $\sqrt{3}$. (h) Centre is (-3, -7) and has a radius of $2\sqrt{2}$.

Question 3: Write the equations of each of these circles.



Purposeful Practice

Question 4: Find if the coordinates given lie on the circles below:

- (a) Does $(4, 2)$ lie on the circle with equation $(x - 1)^2 + (y + 2)^2 = 25$
- (b) Does $(3, 9)$ lie on the circle with equation $(x - 4)^2 + (y - 3)^2 = 36$
- (c) Does $(-16, 4)$ lie on the circle with equation $(x + 4)^2 + (y - 9)^2 = 169$
- (d) Does $(2, -2)$ lie on the circle with radius 8 and centre $(5, 5)$?
- (e) Does $(0, 7)$ lie on the circle with radius $\sqrt{10}$ and centre $(-3, 8)$?

Question 5: Find where the circle $(x - 3)^2 + (y - 5)^2 = 106$ meets the x-axis.

Question 6: Find where the circle $(x - 1)^2 + (y + 2)^2 = 20$ meets the x-axis.

Question 7: Find where the circle $(x + 2)^2 + (y + 3)^2 = 45$ meets the x-axis.

Question 8: Find where the circle $(x - 2)^2 + (y - 1)^2 = 68$ meets the x-axis.

Question 9: Find where the circle $(x - 4)^2 + (y - 6)^2 = 32$ meets the x-axis.

Apply

Question 1: A circle has centre $(7, 2)$.

The point $(1, -6)$ lies on the circle.

Find the equation of the circle.

Question 2: A circle has centre A.

The points C $(0, 4)$ and D $(10, 4)$ lie on the diameter of the circle.

(a) Find the coordinates of A.

(b) Find the equation of the circle.

Purposeful Practice

(a) A circle C has the equation $x^2 + (y + 3)^2 = 169$. The line $x = 5$ passes through the circle at points A and B. Find the length of the line AB.

(b) The points A $(-3, 5)$ and B $(7, 1)$ lie on circle C. The line AB is a diameter of the circle. Find the equation of the circle.

(a) The point P with coordinates $(1, 8)$ lies on the circle with equation $x^2 + y^2 + 4x - 6y - 21 = 0$. Point Q also lies on the circle, and PQ is a diameter of the circle. Find the coordinates of point Q.

(b) A circle C has centre $(-3, -1)$. Point P with coordinates $(3, 2)$ lies on circle C. Find the coordinates of the points where the circle crosses the y -axis.

(a) Determine whether the point $(4, 5)$ lies inside, outside or on the circle with equation $x^2 + y^2 + 4y - 49 = 0$.

(b) A circle has diameter AB where A is $(-5, -1)$ and B is $(0, -7)$. Find the equation of the tangent to the circle at point A, giving your answer in the form $ax + by + c = 0$, where a , b and c are integers to be found.

(a) A circle with equation $x^2 + y^2 = 25$ has centre O and passes through the point P with coordinates $(3, 4)$. Line L is the tangent to the circle at point P. Line L meets the x -axis at A and the y -axis at B. Find the area of the triangle OAB.

(b) The circle C has the equation $x^2 + y^2 - 12x + 4y - 24 = 0$. Find the two values of a for which the line $y = a$ is a tangent to circle C.

Purposeful Practice

Sample Problems

1. Find the coordinates of all points where the circle $x^2 + (y - 3)^2 = 50$ and the line $y = -2x + 8$ intersect each other.
2. a) Find the points where the circles $(x + 4)^2 + (y + 1)^2 = 10$ and $(x + 1)^2 + (y - 5)^2 = 25$ intersect each other.
b) Find the points where the circles $(x + 2)^2 + (y + 2)^2 = 50$ and $(x - 2)^2 + (y - 1)^2 = 25$ intersect each other.
3. Consider the circles $x^2 + y^2 = 36$ and $(x - 5)^2 + y^2 = 16$. Let C be the intersection of the two common tangent lines drawn to the circle.
 - a) Find the coordinates of C .
 - b) Consider one of the common tangent lines. Find the distance between the points of tangency.
4. Find an equation of the circle that passes through the points $A(0, 2)$, $B(-2, -2)$, and $C(-8, -4)$.

Practice Problems

1. Find the coordinates of all point(s) where the circle and the line intersect each other.
 - a) $(x + 5)^2 + (y - 7)^2 = 8$ and $x + y = 6$
 - b) $(x - 2)^2 + (y + 1)^2 = 50$ and $y = -x + 9$
 - c) $x^2 + (y + 4)^2 = 25$ and $y = x - 24$
 - d) $(x - 3)^2 + (y + 1)^2 = 25$ and $y = -x + 1$
2. Find the coordinates of all points where the given circles intersect each other.
 - a) $(x - 1)^2 + (y - 2)^2 = 10$ and $(x - 10)^2 + (y - 5)^2 = 40$
 - b) $(x - 3)^2 + y^2 = 20$ and $x^2 + (y - 1)^2 = 50$
 - c) $x^2 + (y - 8)^2 = 26$ and $(x - 7)^2 + (y - 3)^2 = 4$
3. Consider the circles $(x + 5)^2 + y^2 = 100$ and $x^2 + y^2 = 49$. Let C be the intersection of the two common tangent lines drawn to the circle.
 - a) Find the coordinates of C .
 - b) Consider one of the common tangent lines. Find the distance between the points of tangency.
4. Let C_1 and C_2 be circles defined by $x^2 + (y + 7)^2 = 1$ and $x^2 + (y - 6)^2 = 16$, respectively. Consider one of the common tangent lines. Find the distance between the points of tangency.
5. Given the three points, find an equation of the circle that contains all three points.
 - a) $P(0, 0)$, $Q(0, 8)$, and $R(-6, 0)$.
 - b) $A(-1, 6)$, $B(-7, 12)$, and $C(-3, 0)$.
 - c) $X(3, -7)$, $Y(5, -1)$, and $Z(-3, -5)$.

Purposeful Practice

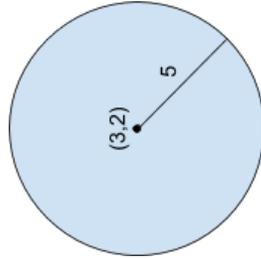
INCREASINGLY DIFFICULT CIRCLE EQUATIONS

1) Find the centre and radius of the circle with the equation $(x - 2)^2 + (y + 1)^2 = 9$.

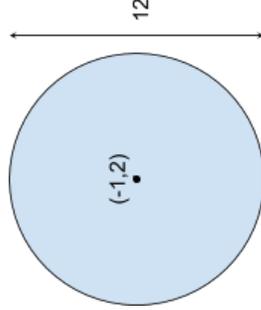
2) Find the centre and radius of the circle with the equation $(x + 2)^2 + y^2 = 40$.

3) Find the centre and radius of the circle with the equation $x^2 + y^2 - 4x + 2y - 4 = 0$.

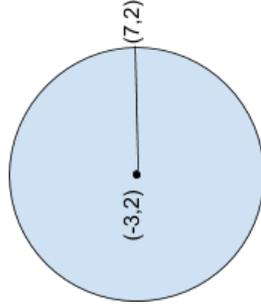
4) Find the equation of the circle:



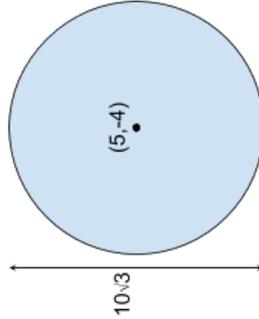
5) Find the equation of the circle:



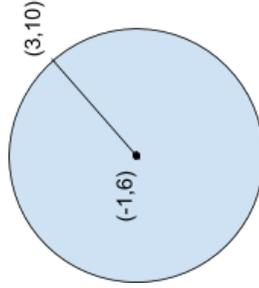
6) Find the equation of the circle:



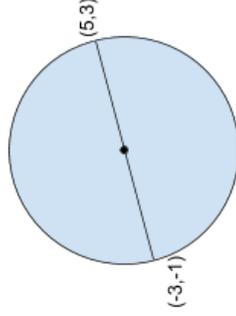
7) Find the equation of the circle:



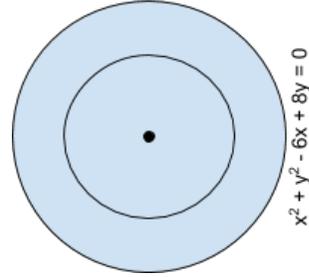
8) Find the equation of the circle:



9) Find the equation of the circle:

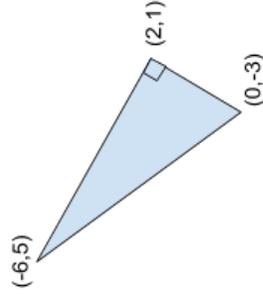


10) Find the equation of the circle which is concentric to $x^2 + y^2 - 6x + 8y = 0$ but is half as wide.

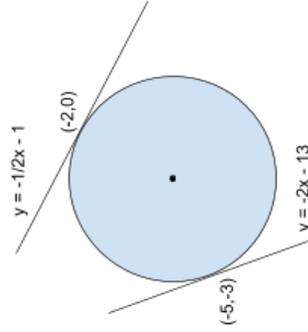


$$x^2 + y^2 - 6x + 8y = 0$$

11) Find the equation of the circle passing through the vertices of the right angled triangle as shown:



12) Find the equation of the circle with a tangent $y = -2x - 13$ which touches the circle at point $(-5, -3)$, and a tangent $y = -\frac{1}{2}x - 1$ which touches the circle at point $(-2, 0)$.



Purposeful Practice

- ① Write down the equations of these circles.
- (i) centre $(1, 2)$, radius 3
 - (ii) centre $(4, -3)$, radius 4
 - (iii) centre $(1, 0)$, radius 5
 - (iv) centre $(-2, -2)$, radius 2
 - (v) centre $(-4, 3)$, radius 1
- ② For each of the circles given below
- (a) state the coordinates of the centre
 - (b) state the radius
 - (c) sketch the circle, paying particular attention to its position in relation to the origin and the coordinate axes.
- (i) $x^2 + y^2 = 25$
 - (ii) $(x - 3)^2 + y^2 = 9$
 - (iii) $(x + 4)^2 + (y - 3)^2 = 25$
 - (iv) $(x + 1)^2 + (y + 6)^2 = 36$
 - (v) $(x - 4)^2 + (y - 4)^2 = 16$
- ③ Work out the equation of the circle with centre $(2, -3)$ which passes through $(1, -1)$.
- ④ A and B are $(4, -4)$ and $(2, 6)$ respectively. Work out
- (i) the midpoint C of AB
 - (ii) the distance AC
 - (iii) the equation of the circle that has AB as its diameter.
- ⑤ Show that the equation $x^2 + y^2 - 4x - 8y + 4 = 0$ represents a circle. Hence give the coordinates of the centre and the radius of the circle, and sketch the circle.
- ⑥ Why does the equation $x^2 - 4x + y^2 - 4y + 22 = 0$ not represent a circle?
- ⑦ A circle of radius 5 cm passes through the points $(0, 0)$ and $(0, 6)$. Sketch two possible positions of the circle and write down the equation in each case.
- ⑧ $(6, 3)$ is a point on the circle with centre $(11, 8)$.
- (i) Work out the radius of the circle.
 - (ii) Work out the equation of the circle.
 - (iii) Work out the coordinates of the other point where the diameter through $(6, 3)$ meets the circle.

Purposeful Practice

- (a) The point P $(6, -2)$ lies on the circle with equation $x^2 + y^2 = 40$. Find the gradient of the tangent to the circle at point P.
- (b) The point Q $(2, 1)$ lies on the circle with equation $x^2 + y^2 = 5$. Find the gradient of the tangent to the circle at point Q.

- (a) The point A $(2, -5)$ lies on the circle with equation $x^2 + y^2 = 29$. Find the equation of the tangent to the circle at point A.
- (b) The point B $(-3, -2)$ lies on the circle with equation $x^2 + y^2 = 13$. Find the equation of the tangent to the circle at point A.

- (a) The point R $(-2, -3)$ lies on the circle with centre $(1, 2)$ and radius $\sqrt{34}$. Find the equation of the tangent to the circle at point R.
- (b) The point S $(7, 1)$ lies on the circle with centre $(x - 3)^2 + (y + 2)^2 = 25$. Find the equation of the tangent to the circle at point S.

- (a) Point P $(-2, 8)$ lies on a circle with centre $(-1, 6)$. Point Q with coordinates $(a, 5)$ lies on the tangent to the circle at P. Find the value of a .
- (b) Point A $(-1, -3)$ lies on the circle with equation $(x - 3)^2 + (y + 2)^2 = 17$. The line L is the tangent to the circle at point A. Find the coordinates of the point where line L crosses the x -axis.

Purposeful Practice

- 1) The circle C has equation $(x - 3)^2 + (y - 7)^2 = 100$.
- Verify the point $P(11,1)$ lies on C .
 - Find an equation of the tangent to C at the point P , giving your answer in the form $ax + by + c = 0$

- 2) A circle C has equation $(x - 4)^2 + (y + 4)^2 = 10$
The line l is a tangent to the circle and has gradient -3 .
Find two possible equations for l , giving your answers in the form $y = mx + c$.

- 3) A circle has centre $C(3,5)$, and goes through the point $P(6,9)$. Find the equation of the tangent of the circle at the point P , giving your equation in the form $ax + by + c = 0$ where a, b, c are integers..

Purposeful Practice

If a diagram is not given, drawing a sketch may help.

- ① AB is a diameter of a circle. P is a point on the circumference of the circle.

A is (2, 8) and P is (4, -2).

Work out the gradient of BP.

- ② A circle has centre C.

RS is a chord of the circle and R is (-1, 6).

Y(2, 3) is a point on RS such that angle CYR = 90°.

Work out the coordinates of S.

- ③ Look at Figure 5.30.

AB is a diameter of the circle.

A is (k, 5), P is (3, 8) and B is (7, 2).

Work out the value of k.

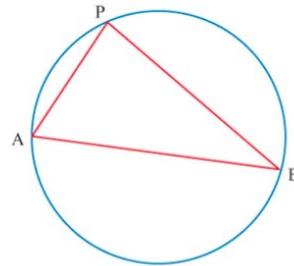


Figure 5.30

- ④ Figure 5.31 shows a circle, centre C.

AB is a chord of a circle.

D is a point on AB such that angle ADC is 90°.

Work out the equation of the line that passes through A and C.

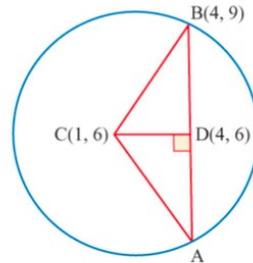


Figure 5.31

- ⑤ Look at Figure 5.32. T(3, -4) is a point on the circumference of the circle $x^2 + y^2 = 25$

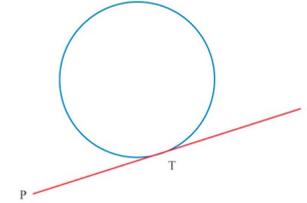


Figure 5.32

Work out the equation of the tangent PTQ.

Give your answer in the form $y = mx + c$.

- ⑥ Figure 5.33 shows a circle that intersects the x-axis at (-2, 0) and (6, 0).

The centre of the circle is (a, 3).

Work out the equation of the circle.

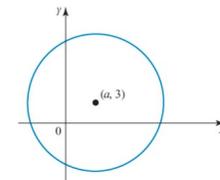


Figure 5.33

- ⑦ Figure 5.34 shows a circle with centre C(2, 1) and radius 5 units together with the tangent at the point A(5, -3).

(i) Write down the equation of the circle.

(ii) Work out the equation of the tangent and verify that the point D(-3, -9) lies on the tangent.

(iii) Write down the equation of the other tangent through the point D(-3, -9) and state the coordinates of the point B where this touches the circle.

(iv) Work out the lengths of the two tangents from D to the circle.

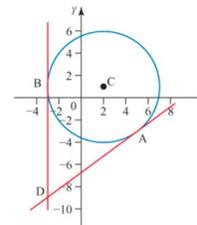


Figure 5.34

- ⑧ Figure 5.35 shows a circle $x^2 + y^2 = 25$ together with the tangent to the circle at the point A(3, -4).

(i) Show that the tangent at A has equation $3x - 4y - 25 = 0$

(ii) Write down the equation of the tangent to the circle at the point B(-5, 0).

(iii) Work out the coordinates of the point of intersection, C, of these tangents.

(iv) Show, by calculation, that $CA = CB$.

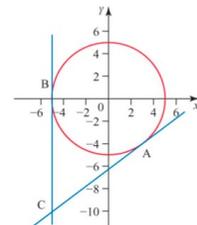


Figure 5.35

3 Advanced Simultaneous Equations

Intelligent Practice

Solve the following pairs of simultaneous equations:

1. $xy = 12$
 $y = x + 1$

5. $xy = 12$
 $y = 2x + 5$

9. $4xy = 12$
 $y = x - 2$

2. $xy = 12$
 $y = x - 1$

6. $xy = 12$
 $y = 2x - 5$

10. $4xy = 12$
 $y = x + 2$

3. $xy = 12$
 $y = x - 4$

7. $xy = 12$
 $2y = x + 5$

4. $xy = 12$
 $y = x + 4$

8. $xy = 12$
 $2y = x - 5$

Intelligent Practice

Solve the following pairs of simultaneous equations:

1. $x^2 + y^2 = 25$
 $y = x + 5$

5. $x^2 + y^2 = 17$
 $y = x + 5$

2. $x^2 + y^2 = 25$
 $y = x - 5$

6. $x^2 + y^2 = 17$
 $y = x - 5$

3. $x^2 + y^2 = 25$
 $y = 2x + 5$

7. $2x^2 + y^2 = 17$
 $y = x + 5$

4. $x^2 + y^2 = 25$
 $y = 2x - 5$

8. $2x^2 + y^2 = 17$
 $y = x - 5$

Intelligent Practice

Solve the following pairs of simultaneous equations:

1. $y = x^2 + 5x - 2$
 $y = x + 3$

5. $y = x^2 - 3x - 2$
 $y = 2x - 8$

2. $y = x^2 - 3x - 2$
 $y = x + 3$

6. $y = x^2 - 3$
 $y = x - 3$

3. $y = x^2 - 5x - 2$
 $y = -x + 3$

7. $y = x^2 - 3$
 $y = x + 3$

4. $y = x^2 + 3x - 2$
 $y = -x + 3$

Fluency Practice

① Solve:

$$y = x^2 + 3x - 1$$

$$y = 2x + 1$$

② Solve:

$$y = x^2 + 5x - 7$$

$$y = 3x + 8$$

③ Solve:

$$xy = 3$$

$$y = x - 2$$

④ Solve:

$$xy = -8$$

$$y = x + 6$$

⑤ Solve:

$$x^2 + y^2 = 13$$

$$y = x + 5$$

⑥ Solve:

$$x^2 + y^2 = 17$$

$$y = x - 3$$

⑦ Solve:

$$x^2 + y^2 = 45$$

$$y = x - 9$$

⑧ Solve:

$$x^2 + y^2 = 5$$

$$y = 2x + 3$$

Purposeful Practice

Solve these simultaneous equations.

(a) $y = x^2 - 4$

$$y = 3x$$

(b) $y = x^2 + 5x$

$$y = 2x + 10$$

(c) $y = 2x^2 + x - 3$

$$y = 3x + 1$$

Solve these simultaneous equations.

(a) $x^2 + 8y = 13$

$$x + 2y = 2$$

(b) $y = 2x^2$

$$3x + y = 20$$

(c) $y = 3x^2 - 4$

$$y = 2x - 3$$

Solve these simultaneous equations.

(a) $x^2 + y^2 = 25$

$$x + y = 7$$

(b) $x^2 + y^2 = 9$

$$y = x + 3$$

(c) $x^2 + y^2 = 5$

$$y = 3x + 5$$

A netball court has an area of 224 m^2 . If the length were decreased by 1 m and the width increased by 1 m, the area would be increased by 1 m^2 . Find the dimensions of the court.

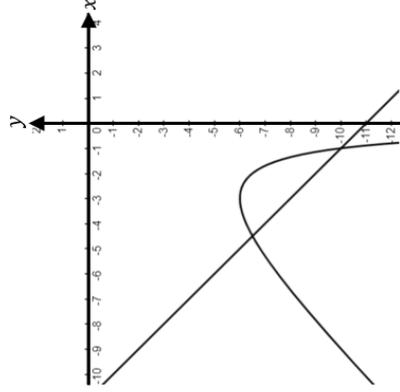
Purposeful Practice

- (a) Solve algebraically $xy = 4$
 $y = x + 3$
- (b) Solve algebraically $xy = -8$
 $x + y = 2$
- (c) Solve algebraically $xy + 12 = 0$
 $x = 2 - 2y$

- (a) Solve algebraically $y = \frac{3}{x}$
 $2x = y - 1$
- (b) Solve algebraically $y = \frac{4}{x} - 2$
 $x = 4y + 8$
- (c) Solve algebraically $y = \frac{4}{x} + x$
 $x + y = 6$

- (a) Solve algebraically $x^2 + xy = 20$
 $x = 4y$
- (b) Solve $x^2 + y^2 - xy = 16$
 $x + y = 4$
- (c) Solve $x^2 - y^2 = 4 + xy$
 $y = 2x - 6$

The diagram shows the graphs of $y = x + \frac{9}{x}$ and $x + y + 11 = 0$. Find the coordinates of the points of intersection.



Fluency Practice

Solve the following simultaneous equations.

1. $x + y = 3$
 $x^2 + y^2 = 5$

2. $x - 4y = 2$
 $3 = 5x - 2y$

3. $2x + y = 1$
 $x^2 + y^2 = 2$

4. $x + 2y = 1$
 $x^2 + 2y^2 = 3$

5. $x - 3y = 2$
 $4x - y = 1$

6. $x - 3y = 2$
 $y^2 - x^2 = -4$

7. $2x + y = 3$
 $4x^2 + y^2 = 5$

8. $2x - 3y = 2$
 $3x + y = 1$

9. $x + 2y = 3$
 $3x^2 + 4y = 6$

10. $2x - y = 4$
 $x^2 + 2y^2 = 17$

11. $x + 3y = 7$
 $x^2 + x + y = 21$

Fluency Practice

Solving Non-Linear Simultaneous Equations			
(a)	(b)	(c)	(d)
Solve $y = x^2 + 2$ $y = 3x$	Solve $y = x^2$ $y = 10 - 3x$	Solve $y = 5 + x^2$ $y = x + 11$	Solve $y = x^2 + 9x - 6$ $y = 5x + 6$
(e)	(f)	(g)	(h)
Solve $y = x^2 - 4$ $x + y = 8$	Solve $y = 6x - x^2$ $y + 5x = 24$	Solve $y = 2x^2 - 1$ $x + y = 5$	Solve $y = x - 3x^2$ $x - y = 3$

Fluency Practice

Q	Solve the simultaneous equations:	Answers
1	$7x + 2y = 29$ and $y = 2x - 2$	
2	$y = x^2 - x - 2$ and $y = 2x - 2$	
3	$x^2 + y^2 = 25$ and $y = 2x - 2$	
4	$7x - 2y = -20$ and $y = x + 5$	
5	$y = x^2 + x + 1$ and $y = x + 5$	
6	$x^2 + y^2 = 13$ and $y = x + 5$	
7	$3x + 4y = 11$ and $y = 3x + 14$	
8	$y = 2x^2 + 3x - 4$ and $y = 3x + 14$	
9	$x^2 + y^2 = 34$ and $y = 3x + 14$	
10	$3x - 2y = -5$ and $y = \frac{1}{2}x - \frac{1}{2}$	
11	$y = -2x^2 - 5x + 1$ and $y = \frac{1}{2}x - \frac{1}{2}$	
12	$(x - 1)^2 + (y + 3)^2 = 17$ and $y = \frac{1}{2}x - \frac{1}{2}$	
13	$ax + by = c$ and $y = dx + e$	
14	$y = ax^2 + bx + c$ and $y = dx + e$	
15	$x^2 + y^2 = c$ and $y = dx + e$	

Fluency Practice

A1 Solve

$$y = x^2$$
$$y = 11x - 10$$

A2 Solve

$$y = x^2$$
$$y + 3x = 4$$

A3 Solve

$$y = 3x^2$$
$$y = 5x + 8$$

A4 Solve

$$y + 7x = 15$$
$$y = 2x^2$$

B1 Solve

$$x^2 + y^2 = 29$$
$$y = 3x + 1$$

B2 Solve

$$x^2 + y^2 = 41$$
$$y = 2x - 3$$

B3 Solve

$$x^2 + y^2 = 17$$
$$y + 2x = 7$$

B4 Solve

$$3x = 4 - y$$
$$x^2 + y^2 = 8$$

C1 Solve

$$y = (x + 1)^2$$
$$y = 5x - 1$$

C2 Solve

$$y = x^2 + 7x + 3$$
$$y = 9x + 2$$

C3 Solve

$$x + y = 11$$
$$xy = 30$$

C4 Solve

$$xy = 6$$
$$2x + y = 7$$

D1 Solve

$$3x - y = 4$$
$$x^2 + y^2 = 34$$

D2 Solve

$$x^2 - y^2 = 8$$
$$3x = y + 8$$

D3 Solve

$$2x^2 - xy = 12$$
$$x = 4 + y$$

D4 Solve

$$3y^2 + 2x^2 = 21$$
$$y - 2x = -7$$

Fluency Practice

Question 1: Solve the following simultaneous equations



(a) $y = x + 3$
 $y = x^2 + 5x - 2$

(b) $y = x^2 + x - 14$
 $y = x - 5$

(c) $y = 2x - 1$
 $y = x^2 - 2x + 2$

(d) $y = 2x^2 + 9x + 1$
 $y = 3x + 9$

(e) $y = 2x^2 + x + 1$
 $y = x^2 - 5x - 7$

(f) $y = -x^2 + 5x + 2$
 $y = 3x^2 - x - 2$

Question 2: Solve the following simultaneous equations



(a) $x + y = 4$
 $y = x^2 + 3x - 1$

(b) $x + y = 7$
 $xy = 10$

(c) $x^2 + y^2 = 13$
 $x + y = 5$

(d) $2x - y + 4 = 0$
 $y = x^2 + x - 2$

(e) $x^2 + y^2 = 29$
 $7 + x + y = 0$

(f) $xy = -6$
 $x + 2y = -4$

(g) $y = x^2 - 3x + 3$
 $y = 10x - 39$

(h) $x^2 + y^2 = 1$
 $x + 2y = 1$

(i) $5x + y = 5$
 $2x^2 - 9x - y = 11$

(j) $y = 9x^2 + 11x + 3$
 $5x - y + 2 = 0$

(k) $2x + y = 5$
 $2x^2 + y^2 = 11$

(l) $x - 4y + 1 = 0$
 $x^2 - 4xy + y^2 = 13$

Question 3: Solve the following simultaneous equations



(a) $2x + y = 7$
 $x^2 - y^2 = 8$

(b) $x^2 + y^2 = 20$
 $y = x + 3$

(c) $y = x^2 - 9x - 3$
 $y = x$

(d) $2x^2 + y^2 = 10$
 $2x - y = 5$

(e) $y = x^2 + x - 7$
 $4x + 2y + 1 = 0$

(f) $y = x - 2$
 $2x^2 - xy = 11$

Purposeful Practice

Apply

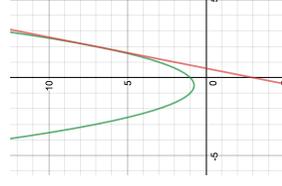
Question 1: Find the coordinates where the line $x + y = 3$ and the curve $x^2 + 3y = 27$ intersect
line $y = x - 8$?

Question 2: How many points of intersection does the curve $y = (x - 3)(x + 4)$ have with the line $y = x - 8$?

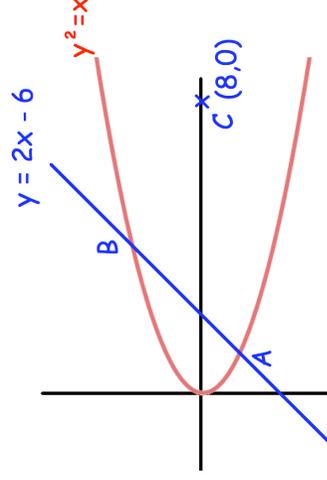
Question 3: The line $y = x + 4$ and the curve $y = x^2 + 3x + 4$ intersect at the points A and B.
Find the distance between the points A and B.

Question 4: Find the coordinates of the points where the line $x + 5y = 37$ and the curve $y = x^2 + x + 2$ meet.

Question 5: Show, by using simultaneous equations, that the line $y = 5x - 3$ is a tangent to the curve $y = x^2 + x + 1$

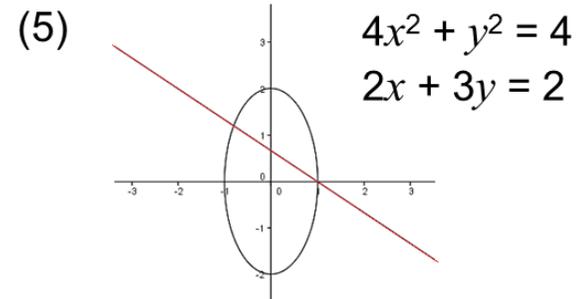
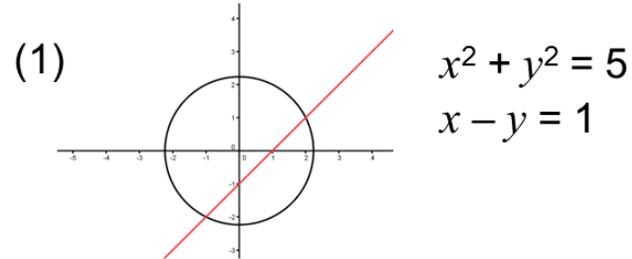


Question 6: Shown below is the curve $y^2 = x$ and the line $y = 2x - 6$
The curve and the line meet at the points A and B
The point C is $(8, 0)$
Show ABC is a right angled triangle.



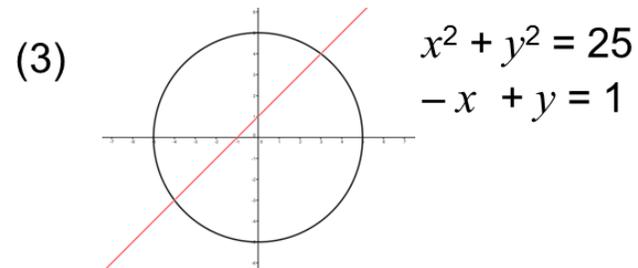
Purposeful Practice

solving simultaneous equations: one linear, one quadratic (i)
find the coordinates of the points of intersection



(2) $x^2 + y^2 = 58$
 $y - x = 4$

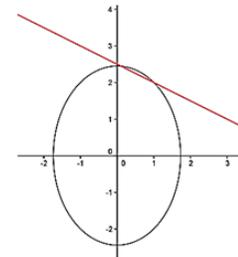
(6) $2x^2 + xy + 1 = 0$
 $y - x = 4$



(7) $4x^2 + y^2 = 37$
 $2y + x = 1$

(4) $x^2 + y^2 = 2$
 $x + 3y = 2$

(8) $2x^2 + y^2 = 6$
 $x + 2y = 5$



factorise any quadratic equations ...

Purposeful Practice

solving simultaneous equations: one linear, one quadratic (ii)

find the coordinates of the points of intersection

$$(1) \quad \begin{aligned} y &= 3x^2 - 14x - 5 \\ y &= 4x - 32 \end{aligned}$$

$$(2) \quad \begin{aligned} x^2 + 8y &= 13 \\ 2y + x &= 2 \end{aligned}$$

$$(3) \quad \begin{aligned} 4x^2 + y^2 &= 10 \\ 2x - y &= 4 \end{aligned}$$

$$(4) \quad \begin{aligned} 2x^2 + y^2 &= 57 \\ 2y + x &= 6 \end{aligned}$$

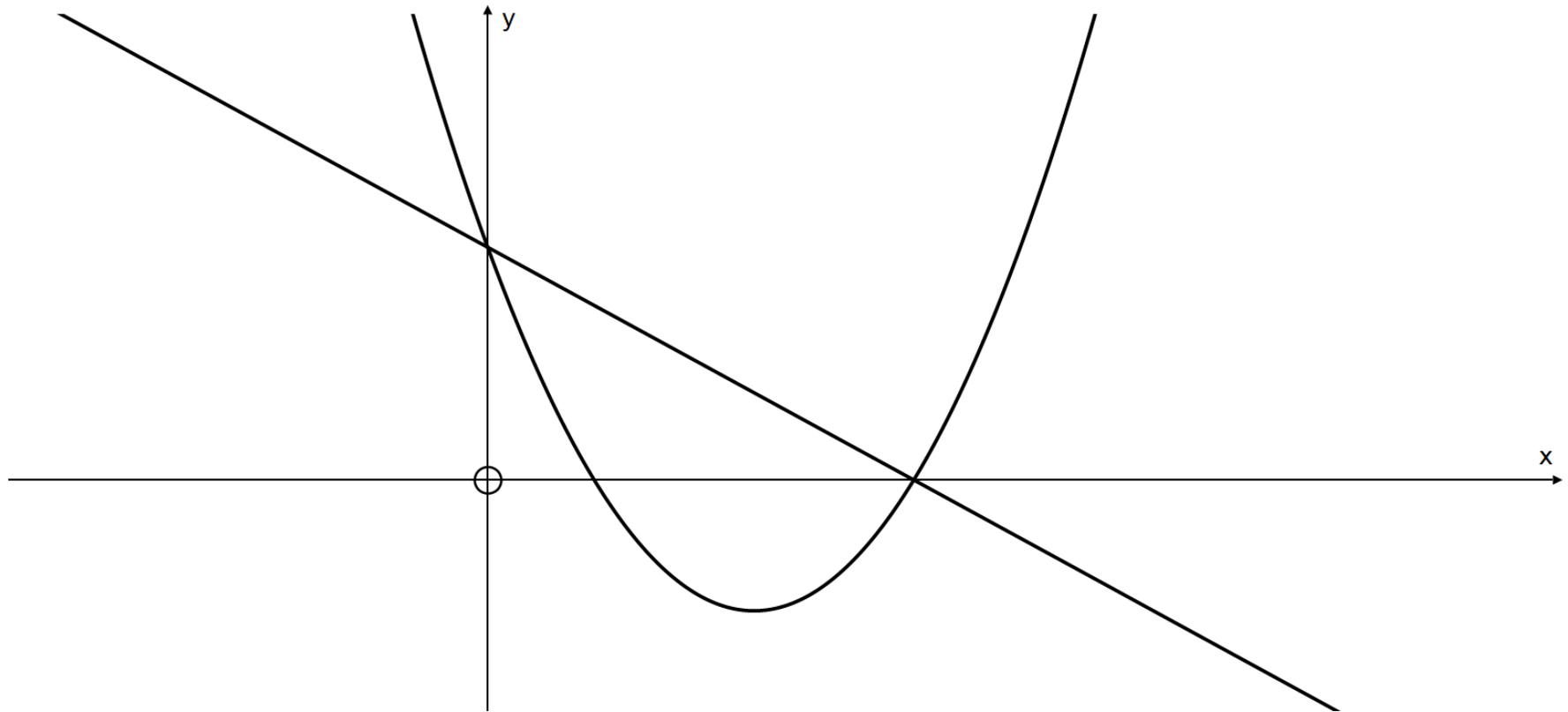
$$(5) \quad \begin{aligned} 2y^2 + x^2 &= 48 \\ y + x &= 8 \end{aligned}$$

$$(6) \quad \begin{aligned} x^2 + y^2 + 2x - 4y &= 8 \\ x + y &= 6 \end{aligned}$$

$$(7) \quad \begin{aligned} x^2 + y^2 + 6x - 4y &= 4 \\ y &= 3x + 4 \end{aligned}$$

$$(8) \quad \begin{aligned} 2x^2 - y^2 + xy &= 14 \\ 4x + 5y &= 0 \end{aligned}$$

Problem Solving



Suggest possible equations for the line and the parabola.

Justify your answer.

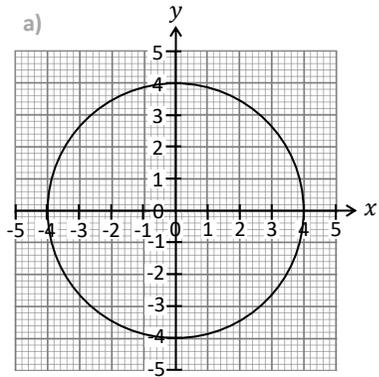
Generalise your answer.

Fluency Practice

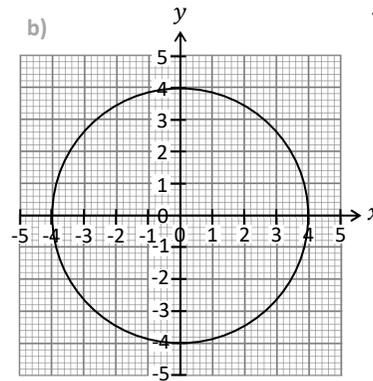
Solving Linear & Non-Linear Simultaneous Equations Graphically

Estimate solutions to each pair of simultaneous equations.

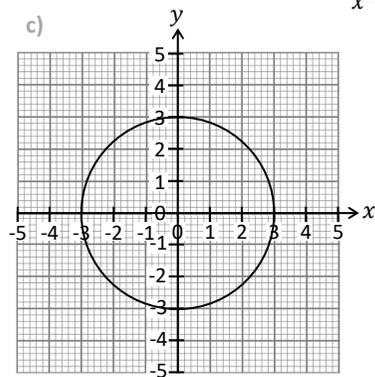
Substitute the linear equation into the non-linear equation to express a quadratic equation in the general form: $ax^2 + bx + c = 0$



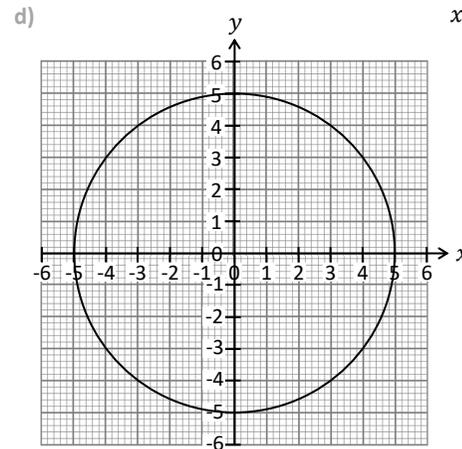
$$x^2 + y^2 = 16$$
$$y = 2x$$



$$x^2 + y^2 = 16$$
$$y = x + 1$$



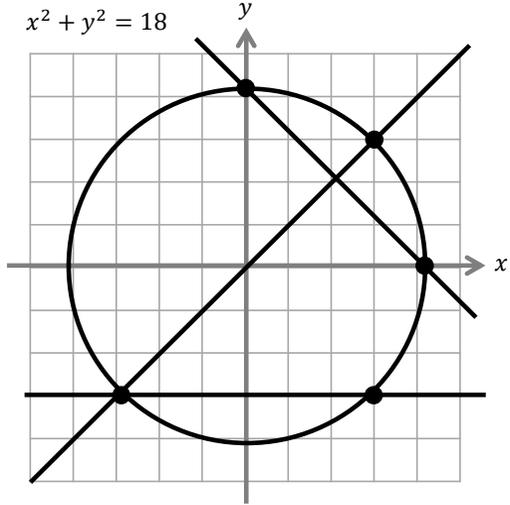
$$x^2 + y^2 = 9$$
$$x + y = 1$$



$$x^2 + y^2 = 25$$
$$y = 3x - 2$$

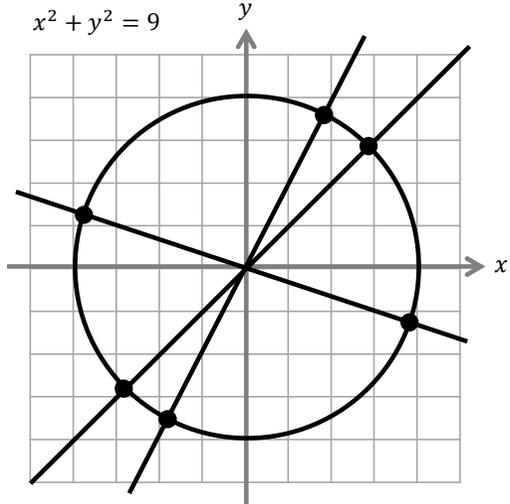
Fluency Practice

$$x^2 + y^2 = 18$$



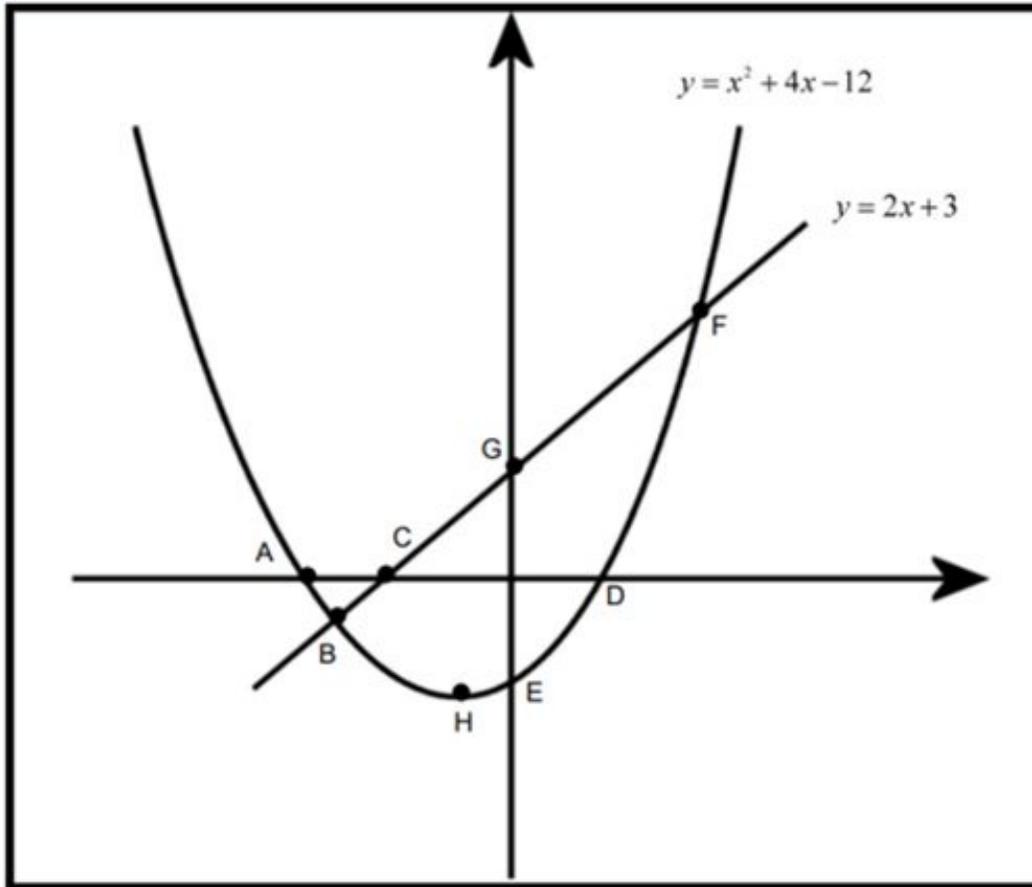
Each grid shows **3 simultaneous equations** involving **1 linear graph & 1 non-linear graph**.
Calculate **exact solutions** for each. Can you substitute & simplify the equations?

$$x^2 + y^2 = 9$$

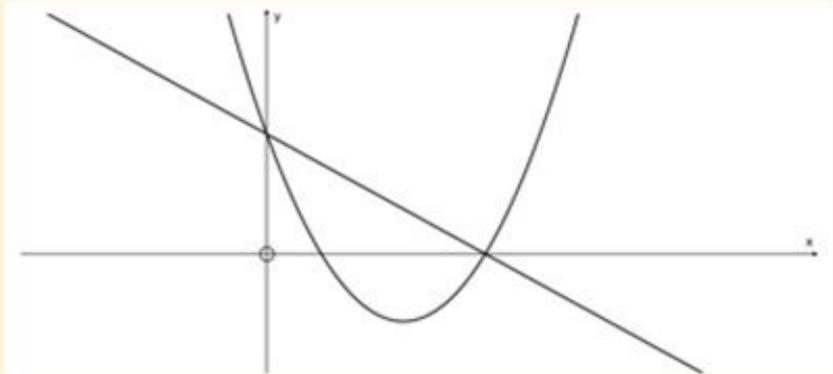


Problem Solving

Find the coordinates of the points A, B, C, D, E, F, G and H



When you finish the task on the left, look at the diagram below. Suggest possible equations for the line and parabola. Justify your answer. Generalise your answer.



Purposeful Practice

- ① Max wants to spend £100 on plants.
But each plant costs £5 more than expected, so
Max has to buy 10 less than planned.
How many plants did Max plan to buy?

- ② **Changing Prices**
Hae wants to spend £90 on plant pots.
But each pot costs £3 more than expected, so
Hae has to buy 5 less than planned.
How many pots did Hae plan to buy?

- ③ Jia wants to spend £160 on calculators.
But each calculator costs £2 more than expected, so
Jia can afford 4 less than they planned.
How many calculators can Jia now buy?

- ④ A rectangle has an area of 84 m^2 .
The length is increased by 1 metre.
The width is increased by 1 metre.
The new area is 104 m^2 .
What was the width of the original rectangle?

- ⑤ **Changing Values**
A rectangle has an area of 60 m^2 .
The length is decreased by 2 metres.
The width is increased by 3 metres.
The new area is 72 m^2 .
What was the length of the original rectangle?

- ⑥ With £360 a shop can buy q t-shirts.
With a price decrease of £8, the shop can
now buy 12 more t-shirts.
How many t-shirts can the shop now buy?

Purposeful Practice

Speed Change

Tim runs 300 metres in t seconds.
By increasing their speed (s) by 1 m/s,
Tim runs the 300 metres 10 seconds faster.

What was Tim's original 300 m time?

$$s = \frac{d}{t} \quad st = d$$

Original: $st = 300$

New: $(s + 1)(t - 10) = 300$

Jan runs 200 metres in t seconds.
By increasing their speed by 3 m/s,
Jan runs the 200 metres 15 seconds faster.

What was Jan's original 200 m time?

A cyclist travels 1 km in t seconds.
Their speed decreases by 5 m/s and they
travel the next kilometre 10 seconds slower.
How long did it take the cyclist to cover the 2 km?

Fluency Practice

Harder Simultaneous Equations			
(a)	(b)	(c)	(d)
Solve $2x + 3y - 18 = 0$ $x = y + 4$	Solve $3x + 4y = 8$ $6 - x = 2y$	Solve $y = x^2 - 2x + 6$ $y = x + 4$	Solve $x^2 + y^2 = 50$ $y = x - 8$
(e)	(f)	(g)	(h)
Solve $x^2 - 5x = y - 5$ $2x + y = 5$	Solve $x^2 + 2y^2 = 22$ $3x = 2y$	Solve $x^2 + y^2 + xy = 12$ $x = 6 - 2y$	Solve $y = x^2 + 3x - 5$ $x - y = 4$
(i)	(j)	(k)	(l)
Find the coordinates of the points where the curve $y = 2x^2 - 3x - 4$ intersects with the line $y = 2x - 1$	Solve $xy = 16$ $x + y = 10$	Solve $x + 2y = 5$ $(x - 1)^2 + (y - 2)^2 = 20$	Find the length of the line joining the points of intersection of $y = \frac{x}{2} + 1$ and $x^2 + y^2 = xy + 4$

Fluency Practice

More Quadratic Equations and Inequalities Revision			
(a)	(b)	(c)	(d)
Solve $x^2 + x - 20 = 0$	Solve, by completing the square, $x^2 + 6x + 3 = 0$ giving answers in surd form	Solve $(x + 3)(x - 5) > 0$	Solve $3x^2 - 11x - 7$ giving your answers to 3 significant figures
(e)	(f)	(g)	(h)
Solve $x^2 \leq 25$	Solve $5x^2 + 18x = 8$	Solve $y = x + 1$ $y = x^2 + 5x - 11$	Solve $2x^2 - 13x + 10 < 0$
(i)	(j)	(k)	
The area of a triangle with base $(x + 9)$ cm and height $(2x + 1)$ cm is 21 cm^2 . Find the value of x .	A rectangle has a length of $(x - 6)$ cm and a width of $4x$ cm. The area of the rectangle is less than 13 cm^2 . Find the range of possible values of x .	Solve $x + 2y = 7$ $x^2 + y^2 = 10$	

4 Advanced Sequences

Fluency Practice

irrational sequences: what comes next?

Look at these sequences, can you work out the term-to-term rule?
What will be the next number in the sequence?

a $\sqrt{2}, 2, 2\sqrt{2}, 4, \dots$

b $\sqrt{3}, 6, 12\sqrt{3}, 72, \dots$

c $\sqrt{2}, \sqrt{8}, \sqrt{18}, \sqrt{32}, \dots$

d $\sqrt{75}, \sqrt{48}, \sqrt{27}, \sqrt{12}, \dots$

e $9, 3\sqrt{3}, 3, \sqrt{3}, \dots$

f $2\sqrt{5}, -10, 10\sqrt{5}, -50, \dots$

g $\sqrt{3}-2, 3-2\sqrt{3}, 3\sqrt{3}-6, 9-6\sqrt{3}, \dots$

h $\sqrt{2}+1, 2\sqrt{2}+3, 5\sqrt{2}+7, 12\sqrt{2}+17, \dots$

extension:

Make up your own sequence. Can you partner figure out your term-to-term rule?

Fluency Practice

geometric series: what are the constant ratios?

(1) 9	(2) —	(3) —	(4) —	(5) —
—	21	—	—	48
576	—	80	324	—
—	189	—	—	—
—	—	1280	26,244	10,368
294,912	—	—	—	—
...	5,103	—	2,125,764	—
	

(6) 4	(7) —	(8) —	(9) 4,782,969	(10) —
—	27	3000	—	12
—	—	—	—	—
5,324	—	—	177,147	—
—	729	81	—	—
—	—	—	—	192
...

what are the first terms (a) and the ratios (r) of these geometric series?

(10) —	(11) —	(12) —	(13) —	(14) —
35	441	1221	200	111
—	—	—	—	—
—	—	—	—	—
—	—	1,625,151	—	—
21,875	1,058,841	...	0.00125	—
...	26,973
				...

Fluency Practice

geometric series: what are the missing terms?

(1) 24
—
54
—
...

(2) —
30
—
480
...

(3) —
—
72
—
32
...

(4) —
—
1200
—
1875
...

(5) —
8
—
—
1

(6) 25
—
—
102.4
...

(7) —
9.9
—
11.979
—
...

(8) —
150
—
216
—
...

(9) 192
—
—
81
...

(10) 250
—
—
54
...

what are 'a' and 'r' for these geometric series?

(10) $ar = 4$
 $ar^4 = 4000$

(11) $ar^2 = 144$
 $ar^6 = 729$

(12) $ar^3 = 1792$
 $ar^6 = 917504$

(13) $ar^2 = 1728$
 $ar^6 = 729$

(14) $ar = 1.6$
 $ar^7 = 0.0001024$

Purposeful Practice

geometric series: find possible values of n and the corresponding constant ratios

(1) $\frac{n-1}{2n}$
 $\frac{3n+9}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

(2) $\frac{n-2}{2n}$
 $\frac{3n+18}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

(3) $\frac{2n-6}{n}$
 $\frac{n-4}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

(4) $\frac{n-2}{2n+2}$
 $\frac{3n+88}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

(5) $\frac{n}{10+n}$
 $\frac{35+2n}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

(6) $\frac{n-2}{2n-2}$
 $\frac{3n+6}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

(7) $\frac{n-1}{n+1}$
 $\frac{3n-1}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

(8) $\frac{n+1}{2n-1}$
 $\frac{5n-7}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

(9) $\frac{2n+2}{3n-1}$
 $\frac{4n-3}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

(10) $\frac{n-2}{2n}$
 $\frac{3n+20}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

(11) $\frac{n+1}{2n+7}$
 $\frac{3n-3}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

(12) $\frac{2n-3}{2-3n}$
 $\frac{3n+8}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

(13) $\frac{3n-2}{4n-2}$
 $\frac{5n-1}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

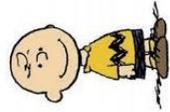
(14) $\frac{n-4}{3n-4}$
 $\frac{10n-4}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

(15) $\frac{3n-2}{3n+3}$
 $\frac{4n}{\quad}$
 $\frac{\quad}{\quad}$
 \dots

Fluency Practice

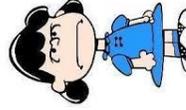
1. Below is the first number in a geometric sequence. Write the first 4 terms of 2 possible sequences that could start with that number:

- | | | | |
|-------|------------------|---------------------|--|
| a. 2 | f. 0.4 | k. ab | |
| b. 4 | g. -4 | l. $\frac{a^2b}{2}$ | |
| c. 8 | h. $\frac{1}{4}$ | m. \sqrt{a} | |
| d. 10 | i. -0.3 | n. ab^2c^3 | |
| e. 7 | j. $\sqrt{2}$ | o. $\frac{\pi}{2}$ | |



2. Calculate the first 5 terms of these sequences:

- | | | | |
|-----------------|---|------------------------------------|--|
| a. 2^n | f. 3×3^n | k. $a \times 3^{bn}$ | |
| b. 2^{n+1} | g. $0.4 \times 3^{(n-1)}$ | l. $\frac{h}{3^n}$ | |
| c. $(-2)^n$ | h. $\frac{2}{5} \times 1.1^n$ | m. $n \times a^n p^2$ | |
| d. $-(-2)^n$ | i. $9.1 \times 10^{\left(\frac{1}{2}\right)^n}$ | n. $m \times 0.03^{(b+n)}$ | |
| e. $0.2^{(2n)}$ | j. $0.03 \times 0.9^{(3n)}$ | o. $\frac{1}{f} \times ad^{(n-3)}$ | |



3. Work out $T(n)$ for the following geometric series:

- | | | | |
|----------------------|--|---|--|
| a. 4, 8, 16, 32 | f. -2, 4, -8, 16 | k. ab, ab^2, ab^3, ab^4 | |
| b. 4, 2, 1, 0.5 | g. 4, -4, 4, -4 | l. $\frac{a}{b^2}, \frac{a^2}{b^3}, \frac{a^3}{b^4}, \frac{a^4}{b^5}$ | |
| c. 5, 15, 45, 135 | h. $\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}$ | m. $-a^2, a^2, -a^2, a^2$ | |
| d. 3, 18, 108, 648 | i. $\sqrt{2}, 2, 2\sqrt{2}, 4$ | n. $-ab, ab^2, -ab^3, ab^4$ | |
| e. -5, -10, -20, -40 | j. $5, 5\sqrt{5}, 25, 25\sqrt{5}$ | o. $a^2b^2, a^2b, a^2, \frac{a^2}{b}$ | |



Fluency Practice

Geometric Sequences			
(a)	(b)	(c)	(d)
Find the next two terms in the sequence 7, 14, 28, 56, ...	Find the next two terms in the sequence 40, 20, 10, 5, ...	Find the first four terms of the sequence with first term 2 and common ratio 3	Find the first term and common ratio for the sequence: 3, 15, 75, 375, ...
(e)	(f)	(g)	(h)
Find the first term and common ratio for the sequence: 160, 80, 40, 20, ...	Find the next two terms in the sequence 2, -4, 8, -16, ...	Find the first four terms of the sequence with first term 120 and common ratio 0.5	Find the first term and common ratio for the sequence: 4, -8, 16, -32, ...
(i)	(j)	(k)	(l)
Find the first four terms of the sequence with first term 5 and common ratio -2	Find the first four terms of the sequence with nth term $6 \times 3^{n-1}$	Find the nth term of the sequence with first term 10 and common ratio 4	Find the nth term of the sequence with first term 250 and common ratio 0.2
(m)	(n)	(p)	
Find the first four terms of the sequence with nth term $400 \times \left(\frac{1}{2}\right)^{n-1}$	Find the nth term of the sequence with first term 8 and common ratio -5	A tree starts with four branches. Every month each branch splits into two. How many branches will the tree have after 5 months? Find a formula for the number of branches b after n months.	

Fluency Practice

Q	Find the value(s) of x where...	Answers
1	$x - 1, 2x + 5$, and $4x + 8$ are three consecutive terms of an arithmetic sequence	
2	$x - 1, 2x + 5$, and $4x + 8$ are three consecutive terms of a geometric sequence	
3	$x - 4, x + 2$ and $3x + 1$ are three consecutive terms of an arithmetic sequence	
4	$x - 4, x + 2$ and $3x + 1$ are three consecutive terms of a geometric sequence	
5	$x - 3, x + 1$ and $4x - 2$ are three consecutive terms of an arithmetic sequence	
6	$x - 3, x + 1$ and $4x - 2$ are three consecutive terms of a geometric sequence	
7	$x - 1, x + 4, 2x + 7$ are three consecutive terms of an arithmetic sequence	
8	$x - 1, x + 4, 2x + 7$ are three consecutive terms of a geometric sequence	
9	$\sqrt{x} - 1, 1$ and $\sqrt{x} + 1$ are three consecutive terms of an arithmetic sequence	
10	$\sqrt{x} - 1, 1$ and $\sqrt{x} + 1$ are three consecutive terms of a geometric sequence	

Fluency Practice

By finding the first and second differences, decide whether each of these sequences is quadratic.

- (a) 1, 5, 11, 19, 29, 41
- (b) 2, 5, 8, 11, 14, 17
- (c) 0, 8, 22, 41, 68, 98
- (d) 2, 9, 20, 35, 54, 77
- (e) 4, 1, 0, 1, 4, 9
- (f) 6, 17, 36, 65, 98, 141
- (g) 18, 37, 62, 93, 130, 173
- (h) 3, 9, 23, 43, 75, 113
- (i) -10, -4, 12, 38, 74, 120
- (j) 17, 39, 69, 107, 153, 207

Generate the first five terms of each of these quadratic sequences.

- (a) $n^2 + 10$ (b) $n^2 - 1$
- (c) $n^2 + n$ (d) $n^2 + 2n + 1$
- (e) $n^2 - 3n$ (f) $n^2 - n - 2$
- (g) $2n^2 + 5$ (h) $3n^2 - 7$
- (i) $2n^2 + n - 5$ (j) $4n^2 + 3n - 1$

Find the 10th and 50th term of the following quadratic sequences.

- (a) $n^2 + 5$ (b) $n^2 - 2$
- (c) $n^2 - n$ (d) $n^2 + 2n$
- (e) $n^2 - 3n + 1$ (f) $n^2 - n - 2$
- (g) $4n^2 + 1$ (h) $3n^2$
- (i) $2n^2 + n - 1$ (j) $5n^2 + 3n$

Fluency Practice

QUADRATIC SEQUENCES Substitute to complete each term.

	1 st	2 nd	3 rd	4 th	5 th	10 th
n^2				16		
$n^2 - 5$				11	20	
$n^2 + n$		6	12			
$n^2 + 2n$	3				35	
$n^2 - n$				12	20	
$n^2 + n + 3$			15	23		
$n^2 + 4n$	5				45	
$n^2 - 3n$				4		
$n^2 + 2n + 6$		14	21			
$n^2 - n + 5$		7			25	
$n^2 + 3n - 4$			14	24		
$n^2 - 2n + 5$	4	5				
$2n^2 + 3n$				44	65	
$2n^2 - 4n$				16		
$2n^2 + 2n + 3$			27		63	
$4n^2 - 3n - 5$		5		47		

Fluency Practice

Find the n th term of these sequences:

- (a) 8, 11, 16, 23, ...
- (b) $-4, -1, 4, 11, \dots$
- (c) 6, 12, 22, 36, ...
- (d) $-1, 5, 15, 29, \dots$
- (e) 4, 13, 28, 49, ...
- (f) 2, 14, 34, 62, ...

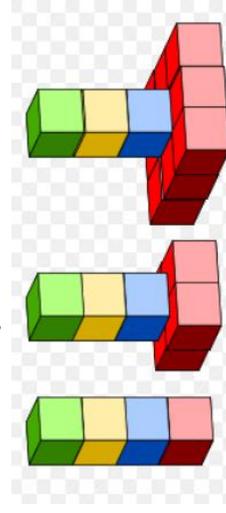
Find the n th term of these sequences:

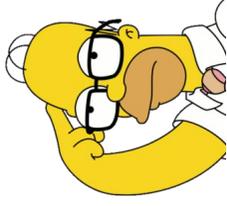
- (a) 3, 7, 13, 21, ...
- (b) 2, 7, 14, 23, ...
- (c) 9, 18, 31, 48, ...
- (d) 1, 5, 13, 25, ...
- (e) 7, 23, 49, 85, ...
- (f) 7, 15, 29, 49, ...

Find the n th term of these sequences:

- (a) 10.5, 12, 14.5, 18, ...
- (b) 5.5, 9, 13.5, 19, ...
- (c) 1.5, 5, 11.5, 21, ...
- (d) 3.5, 10, 19.5, 32, ...
- (e) 19, 16, 11, 4, $-5, \dots$
- (f) 5, 6, 5, 2, $-3, \dots$

Here are some patterns made from cubes. Find an expression for the n th term of this sequence.





Quadratic Sequences

Quadratic sequences take the form

$$an^2 + bn + c$$

For each of the following quadratic sequences, identify the values of a, b and c:

1. $2n^2 + 3n + 4$

4. $n^2 + n - 2$

2. $4n^2 + 8n + 5$

5. $\frac{n^2}{2} - 4$

3. $5n^2 - 11n$

6. $-n^2$

Section 1

Find the first 3 terms for these quadratic sequences:

7. $n^2 + 3$

10. $4n^2 - 30$

13. $\frac{n^2}{2} + 3n + 9$

8. $2n^2 + 2$

11. $n^2 + n - 9$

14. $n(n + 3)$

9. $3n^2$

12. $2n^2 - 2n - 4$

15. $(n + 3)(2n + 6)$

Section 2

Find the 10th and 100th term of these sequences:

16. $n^2 + 3$

18. $-8n^2$

17. $2n^2 - 4n$

19. $\frac{n^2}{3} + 20$

Which of these terms are **not** in the sequences provided in **bold**::

20. {12, 39, 55, 103} (**$n^2 + 3$**)

22. {108, 116, 128, 162} (**$2n^2$**)

21. {15, 35, 63, 82} (**$n^2 - 1$**)

23. {1, 38, 56, 119} (**$n^2 + 10n$**)

Match the sequences to the descriptions below (assume n is always a whole number):

24. $n^2 + 1$

A: Always even

25. $n^2 + n$

B: Always odd

26. $n^2 \div 3$

C: Sometimes even, sometimes odd

27. $2n^2$

D: Sometimes an integer

28. $n^2 + 3n + \frac{3}{5}$

E: Never an integer

Section 3

Find the nth term of these quadratic sequences:

29. 1, 4, 9, 16

34. 0, 6, 16, 30

39. 3, 7, 13, 21

30. -3, 0, 5, 12

35. 8, 14, 24, 38

40. 5, 13, 25, 41

31. 3, 12, 27, 48

36. 11, 20, 35, 56

41. 7, 16, 27, 40

32. 2, 5, 10, 17

37. 0.5, 2, 4.5, 8

42. 0, 1, 4, 9

33. 8, 11, 16, 23

38. 9.3, 10.3, 12, 14.3

43. -5, 8, 27, 52

Section 4



Fluency Practice

Find the n th term for each sequence and the specified term.

-1, 5, 15, 29, 47

10th term

5, 14, 29, 50, 77

20th term

3, 6, 11, 18, 27

50th term

5, 11, 21, 35, 53

25th term

6, 17, 34, 57, 86

10th term

3, 18, 41, 72, 111

20th term

Fluency Practice

find the next two terms in these **quadratic** sequences

(1) 6 , 7 , 10 , 15 , 22 , 31 , ? , ?

(2) 11 , 15 , 21 , 29 , 39 , 51 , ? , ?

(3) - 18 , - 14 , - 8 , 0 , 10 , 22 , ? , ?

(4) - 7 , - 8 , - 7 , - 4 , 1 , 8 , ? , ?

(5) - 10 , - 13 , - 14 , - 13 , - 10 , - 5 , ? , ?

Purposeful Practice

two of the terms in these **quadratic** sequences are wrong

(1) 5 , 0 , -1 , -4 , -1 , 0 , 5 , 12 , 21 , 40

(2) 10 , 6 , 4 , 4 , 6 , 12 , 16 , 24 , 32 , 46

(3) 5 , 1 , -1 , -3 , -1 , 5 , 11 , 19 , 29 , 41

(4) 17 , 11 , 3 , -1 , -3 , -3 , -1 , 3 , 11 , 17

(5) -26 , -22 , -20 , -14 , -6 , 4 , 14 , 30 , 46

Purposeful Practice

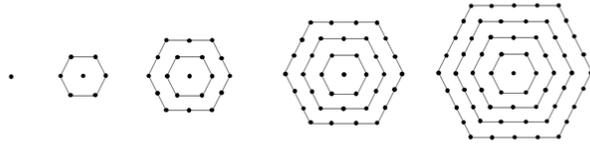
fit a quadratic rule to these sequences

<p>(1) $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \dots \quad n$ $2 \quad 3 \quad 6 \quad 11 \quad 18 \quad 27 \quad \dots$</p>	<p>(6) $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \dots \quad n$ $9 \quad 18 \quad 31 \quad 48 \quad 69 \quad 94 \quad \dots$</p>
<p>(2) $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \dots \quad n$ $3 \quad 6 \quad 11 \quad 18 \quad 27 \quad 38 \quad \dots$</p>	<p>(7) $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \dots \quad n$ $31 \quad 48 \quad 69 \quad 94 \quad 123 \quad 156 \quad \dots$</p>
<p>(3) $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \dots \quad n$ $3 \quad 2 \quad 3 \quad 6 \quad 11 \quad 18 \quad \dots$</p>	<p>(8) $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \dots \quad n$ $4 \quad 9 \quad 18 \quad 31 \quad 48 \quad 69 \quad \dots$</p>
<p>(4) $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \dots \quad n$ $6 \quad 11 \quad 18 \quad 27 \quad 38 \quad 51 \quad \dots$</p>	<p>(9) $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \dots \quad n$ $6 \quad 3 \quad 4 \quad 9 \quad 18 \quad 31 \quad \dots$</p>
<p>(5) $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \dots \quad n$ $11 \quad 18 \quad 27 \quad 38 \quad 51 \quad 66 \quad \dots$</p>	<p>(10) $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \dots \quad n$ $24 \quad 13 \quad 6 \quad 3 \quad 4 \quad 9 \quad \dots$</p>

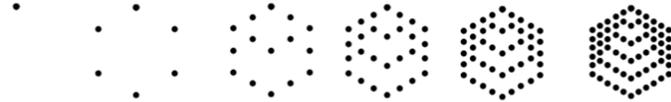
Purposeful Practice

find a rule for the numbers of dots/circles in these quadratic sequences

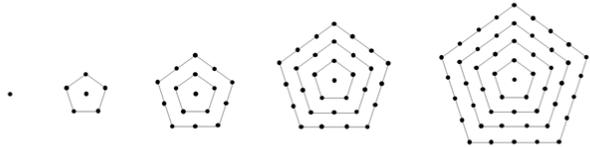
1) centred hexagonal numbers



5) corner hexagonal numbers



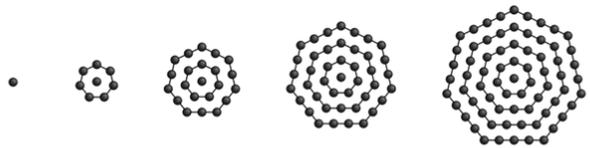
2) centred pentagonal numbers



6) corner pentagonal numbers



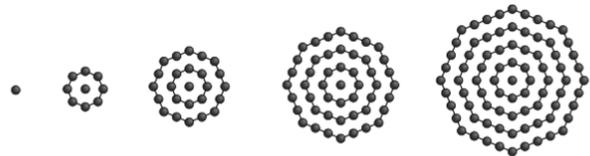
3) centred heptagonal numbers



7) corner heptagonal numbers



4) centred octagonal numbers



8) corner octagonal numbers



Purposeful Practice

The n th term of a sequence is in the form $an^2 + bn$
 The 2nd term in the sequence is 14.
 The 4th term in the sequence is 44.
 Find a & b .

1) Form two equations
 in a & b .

2) Solve the simultaneous equations.

$$\begin{array}{r}
 a(2)^2 + b(2) = 14 \\
 4a + 2b = 14 \xrightarrow{\times 2} 16a + 4b = 28 \\
 a(4)^2 + b(4) = 44 \\
 16a + 4b = 44
 \end{array}
 \qquad
 \begin{array}{r}
 16a + 4b = 44 \\
 \underline{16a + 4b = 28} \\
 8a = 16 \\
 a = 2
 \end{array}
 \qquad
 \begin{array}{r}
 4(2) + 2b = 14 \\
 2b = 6 \\
 b = 3
 \end{array}$$

n th term = $2n^2 + 3n$

Finding the n th Term of a Quadratic Sequence

①

The n th term of a sequence is in the form $an^2 + bn$
 The 2nd term in the sequence is 18.
 The 4th term in the sequence is 52.
 Find a & b .

How can we
 check our answer?

②

A sequence is in the form $an^2 + bn$
 The 3rd term in the sequence is 30.
 The 6th term in the sequence is 114.
 Find the n th term

③

A sequence is in the form $an^2 + bn$
 The 2nd term in the sequence is 22.
 The 3rd term in the sequence is 45.
 Find the n th term

④

The 4th term in a sequence is 64.
 The 5th term in a sequence is 105.
 Find the 3rd term in the sequence.

Purposeful Practice

- (a) A quadratic sequence begins 3, 15, _ , 51, 75,...
What is the the third term?
- (b) The quadratic sequence beginning $-3, 3, 11, 21, 33\dots$
has n th term $n^2 + 3n - 7$.
Find the n th term of the quadratic sequence that begins
3, 11, 21, 33...
- (c) The first term of a quadratic sequence is 10, the fifth term is 25,
and the ninth term 45. Find the thirteenth term of the sequence.

Fluency Practice

Question 1: Find the next two terms for each quadratic sequence

(a) 4, 6, 10, 16, 24 (b) 1, 2, 4, 7, 11 (c) 2, 5, 10, 17, 26

(d) 3, 9, 19, 33, 51 (e) 50, 48, 44, 38, 30 (f) 3, 14, 29, 48, 71

Question 2: List the first 5 terms of the sequences with n^{th} term:

(a) n^2 (b) $n^2 + 1$ (c) $n^2 + 4$ (d) $n^2 - 2$ (e) $2n^2$

(f) $5n^2$ (g) $\frac{1}{2}n^2$ (h) $\frac{1}{4}n^2$ (i) $3n^2 + 10$ (j) $\frac{3}{5}n^2$

Question 3: The quadratic n^{th} term of the sequence below is n^2
1, 4, 9, 16, 25, 36, 49 ...

Find the n^{th} term of each of these sequences

(a) 4, 7, 12, 19, 28, 39, 52 ... (b) 51, 54, 59, 66, 75, 86, 99 ... (c) -5, -2, 3, 10, 19, 30 ...

(d) 3, 12, 27, 48, 75, 108 ... (e) 20, 80, 180, 320, 500, 720 ... (f) 0.2, 0.8, 1.8, 3.2, 5 ...

(g) 3, 9, 19, 33, 51, 73, 99 ... (h) 2.5, 4, 6.5, 10, 14.5, 20 ...

Question 4: For each n^{th} term, work out the first five terms of the sequence.

(a) $n^2 + n$ (b) $n^2 + 2n$ (c) $n^2 - n$ (d) $n^2 - 3n$

(e) $n^2 + n + 2$ (f) $n^2 - 2n + 5$ (g) $n^2 + 4n - 10$ (h) $2n^2 + n$

(i) $3n^2 - n + 6$ (j) $10n^2 + 5n - 7$

Question 5: For each n^{th} term, work out the first five terms of the sequence.

(a) $-n^2$ (b) $-2n^2$ (c) $-4n^2 + 2$ (d) $-n^2 + 3n$

(e) $50 - n^2$ (f) $6n - n^2$ (g) $-n^2 - 7n - 2$

Fluency Practice

Question 6: For each n^{th} term, work out the first five terms of the sequence.

- (a) $n(n+1)$ (b) $n(n+3)$ (c) $(n+1)(n+5)$ (d) $n(n-2)$
(e) $(n-3)(n+1)$ (f) $(n-8)(n-3)$

Question 7: Work out the n^{th} term for each quadratic sequence

- (a) 7, 12, 19, 28, 39 ... (b) 7, 16, 31, 52, 79 ... (c) 6, 13, 24, 39, 58 ...
(d) 3, 13, 27, 45, 67 ... (e) 9, 20, 35, 54, 77 ... (f) 9, 24, 45, 72, 105 ...
(g) -6, -1, 6, 15, 26 ... (h) -5, -4, -1, 4, 11 ... (i) 7, 10, 17, 28, 43 ...
(j) 2.5, 5, 8.5, 13, 18.5 ... (k) -0.5, 1, 4.5, 10, 17.5 ...

Question 8: Calculate the 10th term of each sequence in question 7

Question 9: Work out the n^{th} term for each quadratic sequence

- (a) 3, 1, -3, -9, -17 ... (b) -4, -12, -24, -40, -60 ... (c) 6, 5, 2, -3, -10 ...
(d) 100, 96, 90, 82, 72 ... (e) -17, -30, -49, -74, -105 ... (f) 6, 5.5, 4.5, 3, 1 ...

Question 10: Calculate the 10th term of each sequence in question 9

Question 11: A sequence has an n^{th} term of $n^2 + n - 20$
Work out which term in the sequence has a value of 52.

Question 12: A sequence has an n^{th} term of $n^2 + 2n - 5$
Work out which term in the sequence has a value of 58.

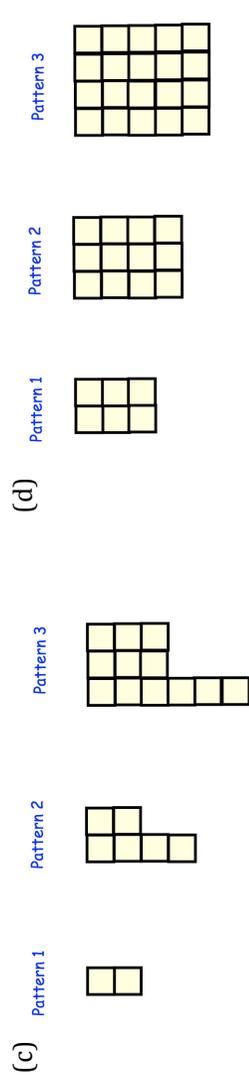
Question 13: A sequence has an n^{th} term of $n^2 - 6n + 7$
Work out which term in the sequence has a value of 23.

Apply

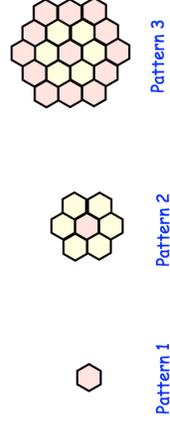
Question 1: The first 5 terms of a quadratic sequence are: 4, 10, 18, 28, 40
Work out the difference between the 10th and 20th terms.

Purposeful Practice

Question 2: Below are patterns of tiles.
The number of tiles in each form quadratic sequences.
Find the number of tiles in pattern n for each.



Question 3: Here is a pattern made from tiles.
How many tiles are needed to make Pattern 20?



Question 4: The first 4 terms of a sequence are: 400, 390, 375, 355 ...
Which term is the first to be negative?

Question 5: The n^{th} term of a quadratic sequence is $n^2 + 4n$
Two consecutive terms have a difference of 25.
Work out the two terms.

Question 6: Prove every term in the sequence $n^2 - 8n + 21$ is positive

Fluency Practice

Q		Answers
1	The n^{th} term of a quadratic sequence is $n^2 + n - 20$. Work out which term has a value of 52.	
2	The n^{th} term of a quadratic sequence is $n^2 + n - 20$. Work out the value of the 52 nd term.	
3	The n^{th} term of a quadratic sequence is $n^2 + 2n - 5$. Work out which term has a value of 58.	
4	The n^{th} term of a quadratic sequence is $n^2 + 2n - 5$. Work out the value of the 58 th term.	
5	The n^{th} term of a quadratic sequence is $n^2 - 6n + 7$. Work out which term has a value of 23.	
6	The n^{th} term of a quadratic sequence is $n^2 - 6n + 7$. Work out the value of the 23 rd term.	
7	The n^{th} term of a quadratic sequence is $2n^2 - 6n + 7$. Work out which term has a value of 7.	
8	The n^{th} term of a quadratic sequence is $2n^2 - 6n + 7$. Work out the value of the 7 th term.	
9	The n^{th} term of a quadratic sequence is $3n^2 - 29n + 75$. Work out which term has a value of 7.	
10	The n^{th} term of a quadratic sequence is $3n^2 - 29n + 75$. Work out the value of the 7 th term.	
11	The n^{th} term of a quadratic sequence is $an^2 + bn + c$. Work out which term has a value of $25a + 5b + c$.	
12	The n^{th} term of a quadratic sequence is $an^2 + bn + c$. Work out the value of the 10 th term	

Fluency Practice

Using the Nth Term of Quadratic Sequences			
$u_n = n^2 + 3n - 5$	$u_n = 3n^2 - n + 1$	$u_n = n^2 - 2n$	$u_n = n^2 + an - b$
(a)	(a)	(a)	(a)
Find the value of u_4	Find the value of u_6 .	Find the 9 th term of the sequence.	Find the value of u_5 in terms of a and b .
(b)	(b)	(b)	(b)
Find the difference between the 6 th term and the 7 th term.	Find the sum of the 9 th term and the 10 th term.	Find an expression for the $(n + 1)^{th}$ term.	Find the value of u_7 in terms of a and b .
(c)	(c)	(c)	(c)
A term of the sequence is 65 Find the value of n .	A term of the sequence is 103 Find the value of n .	Find an expression for the difference between the n^{th} and the $(n + 1)^{th}$ term.	Given that $u_5 = 25$ and $u_7 = 70$, find the values of a and b .

Fluency Practice

Using the Nth Term of Sequences			
$u_n = \frac{6n}{n+7}$	$u_n = \frac{4n+3}{n+1}$	$u_n = \frac{10-3n}{2+n}$	$u_n = \frac{4n^2}{n^2+8}$
(a)	(a)	(a)	(a)
Find the value of u_8 as a fraction in its simplest form.	Find the value of u_9 .	Find the 6 th term.	Find the value of u_5 as a mixed number.
(b)	(b)	(b)	(b)
A term of the sequence is $\frac{11}{3}$ Find the value of n .	A term of the sequence is $\frac{15}{4}$ Find the value of n .	A term of the sequence is $-\frac{7}{5}$ Find the value of n .	Find the term in the sequence closest to 3.8
(c)	(c)	(c)	(c)
Find the difference between the 5 th term and the 9 th term.	Find the sum of the 4 th term and the 14 th term.	Find $2u_{10} - u_{16}$	Find the difference between the 8 th term and the 10 th term.
(d)	(d)	(d)	(d)
Find the first term in the sequence that is greater than 4	Find the first term in the sequence that is greater than 3.9	Find the first term in the sequence that is negative.	Find the smallest value of n for which $u_n > \frac{7}{2}$

5 Limiting Values of Sequences (L2FM Only)

Intelligent Practice

As $n \rightarrow \infty$:

$$\frac{1}{n} \rightarrow$$

$$\frac{7n - 9}{5n - 1} \rightarrow$$

$$\frac{3n + 5}{5n - 9} \rightarrow$$

$$\frac{2}{3n} \rightarrow$$

$$\frac{9 - 7n}{1 - 5n} \rightarrow$$

$$\frac{n^2 + 3n + 5}{5n - 9} \rightarrow$$

$$\frac{2n}{3n} \rightarrow$$

$$\frac{7n^2 - 9}{1 + 5n} \rightarrow$$

$$\frac{6n - 8}{3 + 4n} \rightarrow$$

$$\frac{2n^2}{3n} \rightarrow$$

$$\frac{9 - 7n^2}{1 + 5n} \rightarrow$$

$$2 + \frac{6n - 8}{3 + 4n} \rightarrow$$

$$\frac{7n}{5n + 1} \rightarrow$$

$$\frac{9 - 7n^2}{1 - 5n^2} \rightarrow$$

$$2 - \frac{6n - 8}{3 + 4n} \rightarrow$$

$$\frac{7n + 8}{5n - 1} \rightarrow$$

$$\frac{9 - 7n^2}{1 - 5n^3} \rightarrow$$

$$\frac{2n^2 + 3n - 4}{3n^2 - 4n + 5} \rightarrow$$

Fluency Practice

Find the limits of these sequences as

$$n \rightarrow \infty$$

(a) $\frac{3n}{9n-4}$

(b) $\frac{n}{2n-7}$

(c) $\frac{4n}{2n+1}$

(d) $\frac{8n+1}{2n-3}$

(e) $\frac{3n+2}{6n-4}$

(f) $\frac{2-9n}{15n-4}$

Find the limits of these sequences as

$$n \rightarrow \infty$$

(a) $\frac{n^2}{n^2-1}$

(b) $\frac{4n^2}{n^2+1}$

(c) $\frac{5n^2+1}{10n^2-2}$

(d) $\frac{2n^2-1}{5n^2}$

(a) A sequence starts $\frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \dots$

Find the n th term for this sequence and the limiting value as $n \rightarrow \infty$.

(b) A sequence starts $\frac{1}{5}, \frac{4}{9}, \frac{7}{13}, \frac{10}{17}, \dots$

Find the n th term for this sequence and the limiting value as $n \rightarrow \infty$.

(a) A sequence with n th term $\frac{an+5}{5n-1}$ has a

limiting value of $\frac{2}{5}$ as $n \rightarrow \infty$. Work out the value of a .

(b) A sequence with n th term $\frac{10-bn}{3n+2}$ has

a limiting value of -3 as $n \rightarrow \infty$. Work out the value of b .

Fluency Practice

- ① The n th term of a sequence is $\frac{n+1}{2n+1}$.
- (i) Work out the first three terms of the sequence.
 - (ii) Work out the position of the term that has value 0.52.
- ② The n th term of a sequence is $\frac{4n-1}{2n-5}$.
- (i) Work out the position of the term that has value 2.36.
 - (ii) Show that 1 is not a term in the sequence.

- ③ The n th terms of sequences are shown.
Work out the limiting value of each sequence as $n \rightarrow \infty$.

(i) $\frac{2n}{n+1}$	(ii) $\frac{n+2}{n+3}$	(iii) $\frac{n}{3n-1}$
(iv) $\frac{2n-1}{4n+1}$	(v) $\frac{3n}{4n+1}$	(vi) $\frac{1-n}{n+4}$
(vii) $\frac{2n}{3-4n}$	(viii) $\frac{2-6n}{5-2n}$	

- ④ The n th term of a sequence is $\frac{5n+1}{2n+1}$.
Prove that the limiting value of the sequence as $n \rightarrow \infty$ is $\frac{5}{2}$.
- ⑤ The n th term of a sequence is $\frac{10-6n}{8n-3}$.
Prove that the limiting value of the sequence as $n \rightarrow \infty$ is -0.75
- PS** ⑥ The n th term of a sequence is $2 + \frac{7}{an+b}$, $a \neq 0$
Work out its limiting value.
- PS** ⑦ The n th term of a sequence is $\frac{an+3}{cn-1}$.
The 1st term of the sequence is 11, and its limiting value as $n \rightarrow \infty$ is 4
Work out the value of the 2nd term of the sequence.
- PS** ⑧ The n th term of a sequence is $\frac{2n^2+3n-4}{7n^2-n+2}$.
Work out its limiting value.
(Hint: divide the numerator and denominator by n^2 .)

6 Algebraic Proof

Fluency Practice

Prove It!

a) $2n + 4(n + 1) + 4$
is always even

b) $2(n + 1) + 4(n + 2)$
is always even

c) $5(n + 3) - 3(n - 1)$
is always even

d) $4(2n - 3) - 3(2n - 1)$
is always odd

A number (n)
An even number $2(n)$
An odd number $2(n) + 1$

e) $(n + 1)^2 - n^2 + 1$
is always even

f) $(n + 2)^2 - n^2$
is always even

g) $n^2 + (n + 1)^2$
is always odd

h) $n^2 + (n - 3)^2$
is always odd

i) $(2n + 1)^2 + 3$
is divisible by 4

j) $(n + 1)^2 + (n + 2)^2$
is always odd

k) $(n + 3)^2 - (n + 1)^2$
is divisible by 4

l) $(n - 4)^2 - (n - 2)^2$
is even

m) $(2n + 1)^3$
is always odd

n) $(4n + 1)^2 - (4n - 1)^2$
is divisible by 8

o) $(2n - 1)^2 - 3(n - 2)^2 - 2(n - 10)$
is a square number

Fluency Practice

1. n is a positive integer. Write each expression in the correct box.

- $6n$ $n + 2$ $2n - 3$ $5n$ $4n + 8$ $2n + 1$ $(2n)^2$ $4(n + 1)$ $2n^2 + 4n - 3$ $n^2 + 1$

Always an even number	Sometimes an even number	Never an even number
---------------------------------	------------------------------------	--------------------------------

	Example	Your turn
2.	n is a positive integer. Prove algebraically that $6(n + 3) - n + 2$ is always a multiple of 5 .	n is a positive integer. Prove algebraically that $n(2n + 1) + n^2 + 2n$ is always a multiple of 3 .
3.	n is a positive integer. Prove algebraically that $(n + 2)^2 - n^2$ is always a multiple of 4 .	n is a positive integer. Prove algebraically that $(4n + 1)^2 - 1$ is always a multiple of 8 .
4.	n is a positive integer. Prove algebraically that $(4n + 3)^2 + 2(n + 1)$ is always an odd number .	n is a positive integer. Prove algebraically that $(n + 1)(n + 2) - n^2$ is never a multiple of 3 .

Fluency Practice

expanding quadratic brackets (n is a positive integer)

proofs

(1) prove that $(n + 6)(n + 3)$ is always 10 greater than $(n + 8)(n + 1)$

(2) prove that $(n + 1)(n + 2) - n(n + 3)$ is always 2

(3) prove that $n(n + 1) + (n + 1)(n + 2)$ is double a square number

(4) prove that $(n - 3)(n - 2)$ is always 2 greater than $(n - 4)(n - 1)$

(5) prove that $(3n - 1)(3n + 1) + 1$ is $9 \times$ a square number

(6) prove that $(2n + 1)(2n + 3) + 1$ is an even square number

(7) prove that $(3n + 2)^2$ is always 1 more than $(3n + 1)(3n + 3)$

(8) prove that $(3n + 1)(3n + 2) - 3n(3n + 3)$ is always 2

(9) prove that $(2n + 1)^2 + (n + 3)^2$ is always a multiple of 5

Fluency Practice

1. n is a positive integer. Find a counter-example to show that these statements are **false**:
- a) $3n + 1$ is always even.
 - b) $6n - 1$ is always prime.

	Example	Your turn
2.	Prove algebraically that for any three consecutive integers, the difference between the square of the middle number and the product of the two outer numbers is always 1.	Prove algebraically that the sum of any pair of consecutive odd numbers is always a multiple of 4.
3.	Prove algebraically that the sum of any two odd numbers is an even number.	Prove algebraically that the difference between any two odd numbers is an even number.

4. n is a positive integer. Prove algebraically that $(n + 3)^2 - (n + 1)^2$ is a multiple of 4.

5. Prove that the difference between the squares of two consecutive odd numbers is always a multiple of 8.

Fluency Practice

Algebraic Proof with Multiples		
(a) Show that $4(x + 3) + x - 2$ is always a multiple of 5	(b) Show that $3(7 + 2x) - 9$ is always a multiple of 6	(c) Show that $4(x + 7) + 2(1 - x)$ is always a multiple of 2
(d) Show that $7(2x - 1) - 5(x - 2)$ is always a multiple of 3	(e) Show that $(x + 7)(x - 2) - x^2 - 1$ is always a multiple of 5	(f) Show that $(x + 8)(x + 1) - x(x + 5)$ is always a multiple of 4
(g) Show that $(x + 5)^2 + (x - 3)^2$ is always a multiple of 2	(h) Show that $(3x + 5)(2x - 1) + (2x + 1)(x + 1)$ is always a multiple of 4	(i) Show that $(3x + 2)^2 - (x + 4)(3x - 2) + 4(x - 3)$ is always a multiple of 6

Fluency Practice

- (a) Show that $4(n + 3) - n$ is a multiple of 3 for all integer values of n
- (b) Show that $(n + 2)^2 + 3n^2$ is a multiple of 4 for all integer values of n
- (c) Show that $(3n - 1)^2 - (2n + 1)^2$ is a multiple of 5 for all integer values of n
- (d) Show that $(2n + 1)(4n - 3) - (n + 2)^2 - n$ is a multiple of 7 for all integer values of n

- (a) Show that the sum of three consecutive integers is always a multiple of 3
- (b) Show that the sum of three consecutive even numbers is always a multiple of 6
- (c) Show that the product of two consecutive even numbers is always a multiple of 4

- (a) Prove algebraically that the sum of three consecutive square numbers is never a multiple of 3
- (b) Prove algebraically that the sum of the squares of any two odd numbers is never a multiple of 4
- (c) Prove algebraically that the product of two consecutive odd numbers is never a multiple of 4

- (a) Prove algebraically that the product of three consecutive even numbers is always a multiple of 8
- (b) Prove algebraically that the sum of the cubes of two consecutive even numbers is always a multiple of 8
- (c) Prove algebraically that the product of the squares of two odd numbers is always one more than a multiple of 4

Fluency Practice

Write down algebraic expressions for:

- (a) two different odd numbers
- (b) two consecutive numbers
- (c) two consecutive even numbers
- (d) two different odd numbers squared

- (a) Prove algebraically that the sum of any two odd numbers is always even.
- (b) Prove algebraically that the product of an odd number and an even number is always even
- (c) Prove algebraically that the sum of two consecutive numbers is always odd
- (d) Prove algebraically that the product of two consecutive even numbers is always even

- (a) Prove algebraically that the mean of two consecutive odd numbers is always even
- (b) Prove algebraically that the difference between an odd number and an even number is always odd
- (c) Prove algebraically that the mean of three consecutive odd numbers is always equal to the middle number

- (a) Prove algebraically that the sum of the squares of two even numbers is always a multiple of 4
- (b) Prove algebraically that the difference between the squares of two odd numbers is always a multiple of 4
- (c) Prove algebraically that the sum of the squares of two consecutive numbers is always odd

- (a) Show for all integers values of n that $(n + 3)^2 - n(n - 6) + 2$ is always odd
- (b) Show for all integer values of n that $(3n + 5)^2 + (3 - n)^2$ is always even

Fluency Practice

- (a) Prove that the difference of two odd numbers is even.
- (b) Prove that $n^2 - 2 - (n - 2)^2$ is always an even number.
- (c) Prove that the product of two odd numbers is always odd.
- (d) Prove that the square of an even number is always even.
- (e) Prove that the difference between any two consecutive odd numbers is always two.
- (f) Prove that the mean of three consecutive integers is always the middle number.
- (g) Prove that the difference between the squares of any two consecutive numbers is always odd.
- (h) Prove that $7(n + 8) + 5(n - 4)$ is always a multiple of 12.
- (i) Prove that $(m + 2)^2 - m^2 - 12$ is always a multiple of 4.
- (j) Prove that the sum of three consecutive odd numbers is always a multiple of three.
- (k) Prove that the sum of the squares of any two positive odd integers is always even.
- (l) Prove that $(3n + 1)^2 - (3n - 1)^2$ is always a multiple of 12 for all positive integer values of n .
- (m) Prove that the sum of four consecutive integers is not divisible by four.

Fluency Practice

Q	Prove algebraically that:	Answers. Even $2n/2k$ Odd $2n + 1/2k + 1$
1	the product of two consecutive integers is always even	
2	the sum of two consecutive integers is always odd	
3	the difference between two even numbers is even	
4	the difference between two odd numbers is even	
5	the sum of the squares of any two even numbers is a multiple of 4	
6	the sum of the squares of any two odd numbers is 2 more than a multiple of 4	
7	the sum of three consecutive even integers is even	
8	the sum of three consecutive odd integers is odd	
9	the product of three consecutive even integers is even	
10	the product of three consecutive odd integers is odd	
11	the sum of the squares of any three consecutive even numbers is always 4 less than a multiple of 12	
12	the sum of the squares of any three consecutive odd numbers is always 11 more than a multiple of 12	
13	$(2n + 1)^2 - (2n + 1)$ is an even number for all positive integer values of n	
14	$(3n + 2)^2 - (n - 1)^2$ is an odd number for all positive integer values of n	
15	$n^2 - 2 - (n - 2)^2$ is always an even number for all positive integer values of n greater than 1	
16	$n^2 - 1 - (n - 4)^2$ is always an odd number for all positive integer values of n greater than 1	

Purposeful Practice

triangular number proofs

(1) $1 + 3 = 4$
 $3 + 6 = 9$
 $6 + 10 = 16$
 ...
 general rule? proof?

(2) $3 - 1 = 2$
 $6 - 3 = 3$
 $10 - 6 = 4$
 ...
 general rule? proof?

(3) $1 \times 3 = 3 \times 1$
 $2 \times 6 = 4 \times 3$
 $3 \times 10 = 5 \times 6$
 ...
 general rule? proof?

(4) $8 \times 1 + 1 = 3 \times 3$
 $8 \times 3 + 1 = 5 \times 5$
 $8 \times 6 + 1 = 7 \times 7$
 ...
 general rule? proof?

(5) $3 - 2 \times 1 = 1$
 $10 - 2 \times 3 = 4$
 $21 - 2 \times 6 = 9$
 ...
 general rule? proof?

(6) $3 \times 3 + 1 = 10$
 $3 \times 6 + 3 = 21$
 $3 \times 10 + 6 = 36$
 ...
 general rule? proof?

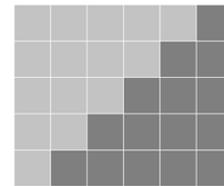
(7) $3 \times 3 + 6 = 15$
 $3 \times 6 + 10 = 28$
 $3 \times 10 + 15 = 45$
 ...
 general rule? proof?

(8) $9 \times 1 + 1 = 10$
 $9 \times 3 + 1 = 28$
 $9 \times 6 + 1 = 55$
 ...
 general rule? proof?

(9) $1^2 + 3^2 = 10$
 $3^2 + 6^2 = 45$
 $6^2 + 10^2 = 136$
 ...
 general rule? proof?

(10) $3^2 - 1^2 = 8$
 $6^2 - 3^2 = 27$
 $10^2 - 6^2 = 64$
 ...
 general rule? proof?

$$T_n = \frac{1}{2} n (n + 1)$$



n	T_n
1	1
2	3
3	6
4	10
5	15
6	21
7	28
8	36
9	45
10	55
11	66
12	78
13	91
14	105
15	120
16	136
17	153
18	171
19	190
20	210
21	231
22	253
23	276
24	300
25	325
26	351
27	378
28	406
29	435

Purposeful Practice

consecutive numbers – setting up and using an equations to solve a problem

find three consecutive numbers that add up to 45

find four consecutive numbers that add up to 38

find five consecutive numbers that add up to 60

any number is n ; the
next number is $n + 1$

any even number is
 $2n$; the next even
number is $2n + 2$ etc

find three consecutive even numbers that add up to 42

find four consecutive even numbers that add up to 60

find five consecutive even numbers that add up to 60

find three consecutive odd numbers that add up to 57

find four consecutive odd numbers that add up to 48

find even numbers of consecutive odd numbers that add up to 72

any odd number is
 $2n + 1$; the next odd
number is $2n + 3$ etc

any multiple of 4 is
 $4n$; the next multiple
of 4 is $4n + 4$ etc

find three consecutive multiples of 3 that add up to 63

find four consecutive multiples of 5 that add up to 150

find five consecutive multiples of 2 that add up to 94

Purposeful Practice

- 1** Write down algebraic expressions for
 - (a) an even number
 - (b) an odd number
 - (c) a multiple of four
 - (d) a positive number which leaves a remainder of 1 upon division by 5
 - (e) the sum of two consecutive even numbers
 - (f) the sum of two even numbers
 - (g) the product of two odd numbers
 - (e) the cube of a multiple of 6
 - (h) the sum of the squares of two consecutive multiples of 3
- 2** Prove algebraically that
 - (a) the sum of two consecutive numbers is odd
 - (b) the product of two even numbers is even
 - (c) the sum of any three consecutive numbers is a multiple of 3
- 3**
 - (a) Prove algebraically that the sum of the squares of two consecutive odd numbers is **not** a multiple of 4.
 - (b) What is the remainder when the sum of the squares of two consecutive odd numbers is divided by 4? Is the number odd or even?
- 4** Johnny has an even number. He adds 3 to this number and then squares it. The result is 81. What even number did Johnny start with?
- 5** Prove algebraically that $(5n + 1)^2 - (5n - 1)^2$ is both a multiple of 4 and a multiple of 5 for positive integers n .
- 6** Anne has a number which is not a multiple of 3.
 - (a) Explain why it must be of the form $(3n + 1)$ or $(3n + 2)$, where n is an integer.
 - (b) Hence, by considering these two different cases, show that the square of Anne's number leaves a remainder of 1 when divided by 3.
- 7**
 - (a) Prove that if n is an even integer, then $3n^2 + n + 14$ is even.
 - (b) Prove that if n is an odd integer, then $3n^2 + n + 14$ is even.
 - (c) What can you deduce about the number $3n^2 + n + 14$ for all integers n ?

Fluency Practice

Negative.... when?

x = any number

Can these expressions have a negative value?

For each expression, test values & *limits* for x .

(hint: factorising the expression, or a part of it may help)

$$x^2$$

$$x^2 - 4$$

$$x^2 + x$$

$$x^2 + 2x$$

$$x^2 - 3x$$

$$x^2 + 2x + 1$$

$$x^2 + 2x + 1$$

Fluency Practice

Question 1: Prove the following

(a) $(n + 7)^2 - (n + 1)^2 = 12(n + 4)$

(b) $(n + 1)^2 - (n - 3)^2 = 8(n - 1)$

(c) $(n + 1)^2 + (n + 5)^2 - (n + 9)^2 = (n + 5)(n - 11)$

Question 2: Prove the following

(a) $(n + 4)^2 - (n + 2)^2$ is always a multiple of 4 for all positive integer values of n .

(b) $(n + 10)^2 - (n + 2)^2$ is always a multiple of 16 for all positive integer values of n .

(c) $(2n + 3)^3 - (2n + 1)$ is always even for all positive integer values of n .

(d) $(5n + 2)^2 - (5n - 1)^2$ is always a multiple of 3 for all positive integer values of n .

(e) $(2n + 9)^2 - (2n + 5)^2$ is always a multiple of 4 for all positive integer values of n .

(f) $(n + 2)^2 - (n - 2)^2 + 3$ is always odd for all positive integer values of n .

Question 3: n is a positive integer.

(a) Write an expression for an even number.

(b) Write an expression of an odd number.

Question 4: Prove the following

(a) The sum of any three consecutive integers is divisible by 3.

(b) The sum of any three consecutive even numbers is always a multiple of 6.

(c) The sum of two consecutive odd numbers is even.

(d) The sum of three consecutive odd numbers is always a multiple of 3.

(e) The sum of four consecutive odd numbers is always a multiple of 8.

(f) The sum of two consecutive integers is always odd.

(g) The sum of four consecutive integers is **not** a multiple of 4

Purposeful Practice

Question 5: Prove the following

- (a) Prove the product of two odd numbers is always odd.
- (b) Prove the product of two even consecutive numbers is always a multiple of 4.
- (c) The difference between the squares of any two consecutive integers is equal to the sum of the two integers.
- (d) Prove the sum of the squares of any two consecutive even numbers is always a multiple of 4.
- (e) Prove that when any odd integer is squared, the result is always one more than a multiple of 8.

Apply

Question 1: The first five terms of a linear sequence are 2, 7, 12, 17, 22

- (a) Find the n th term of the sequence

A new sequence is generated by squaring each term of the linear sequence and then adding 1.

- (b) Prove that all terms in the new sequence are divisible by 5.

Question 2: The first two terms of a fibonacci sequence are a and b .

- (a) Show the 4th term of the sequence is $a+2b$
- (b) Prove that the sum of the first 10 terms is equal to 11 times the 7th term.

Question 3: Cara writes down a 3-digit number where the first digit is greater than the last. e.g. 681

She then reverses the number to give 186.

Cara then subtracts this number from her starting number. $681 - 186 = 495$

She then reverses her answer to give 594.

Cara then adds these number $495 + 594 = 1089$.

$$\begin{array}{r} 681 \\ -186 \\ \hline 495 \\ +495 \\ \hline 1089 \end{array}$$

Cara repeats this several times and always gets 1089 as her answer.
Prove algebraically that the answer is always 1089.

Fluency Practice

$8(ax + 2) \equiv 24x + 16$	$a =$	
$5(2x + a) \equiv 10x + 45$	$a =$	
$3(9x + 1) \equiv 27x + a$	$a =$	
$6(5 - x) \equiv 30 - ax$	$a =$	
$4(7x + 2) + a \equiv 28x + 10$	$a =$	
$5(3x + 4) + ax \equiv 19x + 20$	$a =$	
$2x + 9 + ax - b \equiv 10x + 6$	$a =$	$b =$
$a(7x - 3) \equiv 35x - 15$	$a =$	
$2(ax + 5) + 3(2x + b) \equiv 18x + 13$	$a =$	$b =$
$9(ax + b) + 4(x - 2) \equiv 13x + 37$	$a =$	$b =$

Fluency Practice

Algebraic Proof		
(a)	(b)	(c)
Show that $3x(x + 5) + 2x(x - 5) \equiv 5x(x + 1)$	Show that $(x + 6)(x - 2) + 12 \equiv x(x + 4)$	Show that $(x - 4)^2 + 6x - 16 \equiv x(x - 2)$
(d)	(e)	(f)
Show that $3(8 - x) + 2(5x - 6) \equiv ax + b$ where a and b are integers to be found	Show that $(x + 5)(x - 3) - x(x - 8) \equiv ax + b$ where a and b are integers to be found	Show that $(x + 6)^2 + 4(x - 9) \equiv x(x + a)$ where a is an integer to be found
(g)	(h)	(i)
Show that $(2x + 5)(x - 1) + 3(5 - x) = ax^2 + b$ where a and b are integers to be found	Show that $(x + 4)^2 + (x + 2)(x - 8) = ax(x + b)$ where a and b are integers to be found	Show that $(3x + 4)^2 - (5x + 8)(x + 2) \equiv ax(bx + c)$ where a and b are integers to be found

Fluency Practice

1.	$2(3x + 8) + 6(x - 5) \equiv ax + b$
2.	$10x(x - 3) - 3x(2x - 2) \equiv cx^2 + dx$
3.	$6(2x + 1) + 2(4x - 1) + 10 \equiv ex + f$
4.	$6(x + 3) + g(2x + 10) \equiv 14x + h$
5.	$(x + 7)(x - 6) + 3 \equiv x^2 + jx + k$
6.	$(x + 3)(x + n) + mx^2 + 5x \equiv 4x^2 + 16x + p$
7.	$(x + 5)(x - 2) + qx(2x + 8) \equiv 9x^2 + 35x - 10$
8.	$(x + r)(x - s) + 2x + 15 \equiv x^2 + 8x - 25$

Answers

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>

<i>j</i>	<i>k</i>	<i>m</i>	<i>n</i>	<i>p</i>	<i>q</i>	<i>r</i>	<i>s</i>

Fluency Practice

1	$(2x + 3)(x - 5) + 7 \equiv ax^2 + bx + c$	$a =$ $b =$ $c =$
2	$(x + 1)(x + 4)(x - 1) \equiv x^3 + ax^2 + bx + c$	$a =$ $b =$ $c =$
3	$2x^2 + 12x - 9 \equiv a(x + b)^2 + c$	$a =$ $b =$ $c =$
4	$(ax + 1)(x + c) + b \equiv 5x^2 + 11x + 8$	$a =$ $b =$ $c =$
5	$3(ax + 1) - (6x + b) \equiv 21x - 8$	$a =$ $b =$
6	$(ax + 5)(x - b) + 4x \equiv 2x^2 + cx - 10$	$a =$ $b =$ $c =$
7	$4(3x + a) + 5(ax + b) \equiv 22x + 28$	$a =$ $b =$
8	$4x^2 + ax + 3 \equiv b(x + 1)^2 + c$	$a =$ $b =$ $c =$
9	$\frac{ax^2 + x - b}{x + 2} \equiv x - 1$	$a =$ $b =$
10	$2(5x + 1)(3x + b) + ax(2x + 3) \equiv 44x^2 + 37x + c$	$a =$ $b =$ $c =$

Fluency Practice

Find the value of each letter in the identities below. Circle the answers at the bottom as you find them.

$8(ax + 2) \equiv 24x + 16$ $a = \square$	$5(2x + b) \equiv 10x + 45$ $b = \square$	$c(9x + d) \equiv 18x + 26$ $c = \square, d = \square$	$e(2 + fx) \equiv 24x - 8$ $e = \square, f = \square$
$6(3x + 5) + g \equiv 18x + 44$ $g = \square$	$4(5x - 3) + hx \equiv 13x - 12$ $h = \square$	$i(8x - 3) \equiv 4x + j$ $i = \square, j = \square$	$12(6x + k) - lx \equiv 71x + 4$ $k = \square, l = \square$
$(x + m)(x - 2) \equiv x^2 + 5x - 14$ $m = \square$	$(x + 4)(x - 7) \equiv x^2 + nx - 28$ $n = \square$	$(x + \sigma)(x + 20) \equiv x^2 + px - 160$ $\sigma = \square, p = \square$	$(2x + 1)(x - 5) \equiv 2x^2 + rx + q$ $q = \square, r = \square$
$(x - s)^2 + 10 \equiv x^2 - 10x + 35$ $s = \square$	$(x + 3)^2 - 3 \equiv x^2 + tx + t$ $t = \square$	$(x + u)^2 - v \equiv x^2 + 22x + 71$ $u = \square, v = \square$	$x^2 - 5x + 6 \equiv (x + w)^2 - y$ $w = \square, y = \square$
$\frac{ax - 8}{4x + \beta} \equiv 4$ $\alpha = \square, \beta = \square$	$\frac{x^2 + \gamma x + 44}{x + 11} \equiv x + \delta$ $\gamma = \square, \delta = \square$	$(x + 5)^2(x - 2) \equiv x^3 + \epsilon x^2 + 5x + \zeta$ $\epsilon = \square, \zeta = \square$	$(x - \eta)(x + 8) \equiv (x + \theta)^2 - 81$ $\eta = \square, \theta = \square$

Answers (One of the questions has multiple solutions. Which one? What are the other possible solutions?)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
-1	-2	-3	-4	-5	-6	-7	-8	-9	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$-1\frac{1}{2}$	$-2\frac{1}{2}$	-50	50

Fluency Practice

Find the value of each letter in the identities below. Circle the answers at the bottom as you find them.

$\frac{ax + 12}{4x + 6} \equiv 2$ $a = \square$	$\frac{7x + b}{21x - 12} \equiv \frac{3x + 5}{9x + 15}$ $b = \square$	$\frac{cx - 10}{18x - 12} \equiv \frac{35x - 5}{42x + d}$ $c = \square, d = \square$	$\frac{12 - 9x}{6x + e} \equiv f$ $e = \square, f = \square$
$\frac{x}{3} - \frac{x}{g} \equiv \frac{x}{12}$ $g = \square$	$\frac{x + 1}{7} - \frac{x}{2} \equiv \frac{hx + 20}{140}$ $h = \square$	$\frac{2x + i}{5} + \frac{x + 5}{3} \equiv \frac{jx + 22}{15}$ $i = \square, j = \square$	$\frac{kx - 8}{6} - \frac{4x - 11}{8} \equiv \frac{l}{12}$ $k = \square, l = \square$
$\frac{x^2 + mx}{x^2 + 7x + 10} \equiv \frac{x}{x + 2}$ $m = \square$	$\frac{x^2 - x - 12}{x^2 - 9x + 20} \equiv \frac{x + 3}{x + n}$ $n = \square$	$\frac{x + 4}{x^2 - \sigma} \equiv \frac{p}{x - 4}$ $\sigma = \square, p = \square$	$\frac{x^2 + qx + r}{x^2 + x - 20} \equiv \frac{x^2 + 3x - 18}{x^2 + 11x + 30}$ $q = \square, r = \square$
$\frac{3x^2 - sx - 5}{x - 5} \equiv 3x + 1$ $s = \square$	$\frac{3x^2 + tx}{2x^2 - 18} \equiv \frac{3x}{2x - 6}$ $t = \square$	$\frac{x^2 + ux - 18}{2x^2 - x - 6} \equiv \frac{x + 9}{vx + 3}$ $u = \square, v = \square$	$\frac{wx^2 + 12x - 1}{26x^2 + 11x - 1} \equiv \frac{11x^2 + yx - 1}{22x^2 + 9x - 1}$ $w = \square, y = \square$
$\frac{5}{x + 3} - \frac{4}{x + \alpha} = \frac{x - 27}{x^2 + \beta}$ $\alpha = \square, \beta = \square$	$\frac{\gamma}{x + 4} - \frac{x - 44}{x^2 - 16} \equiv \frac{\delta}{10x - 40}$ $\gamma = \square, \delta = \square$	$\frac{6\zeta}{5 - x} + \frac{3}{x + 7} \equiv \frac{29 - x}{14\epsilon - 2x - x^2}$ $\epsilon = \square, \zeta = \square$	$\frac{1}{x} + \frac{1}{x + 1} + \frac{\eta}{x + 2} \equiv \frac{12\theta x + 2}{x^3 + 3x^2 + 2x}$ $\eta = \square, \theta = \square$

Answers

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
-1	-2	-3	-4	-5	-6	-7	-8	-9	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$-1\frac{1}{2}$	$-2\frac{1}{2}$	-50	50

Fluency Practice

versions

find the values of the letters in these (equivalent) 'families' of quadratic expressions

$$(1) \quad (x + 2)(x + 6) + 3$$

$$(x + 7)(x + a) + b$$

$$(x + c)^2 - d$$

$$(x + 1)(x + 15) - ex$$

$$(x + f)(x + g)$$

$$(2) \quad (x - 8)(x + 1) + 14$$

$$(x - 2)(x - a) - b$$

$$(x - 4)(x - c) - d$$

$$(x - 3)(x - 2) - ex$$

$$(x - f)(x - g)$$

$$(3) \quad (x - 2)(x - 10) + 15$$

$$(x + 3)(x - a) + b$$

$$(x - c)^2 - d$$

$$(x + 5)(x + 7) - ex$$

$$(x - f)(x - g)$$

$$(4) \quad (x - 3)(x - 5) - 35$$

$$(x - 11)(x + a) + b$$

$$(x - c)^2 - d$$

$$x^2 - 4(2x + e)$$

$$(x + f)(x - g)$$

$$(5) \quad (2x + 3)(x + 8) + 11$$

$$(2x + 7)(x + a) - b$$

$$(x + 9)(2x + c) + d$$

$$(x - 5)(2x - 7) + ex$$

$$(2x + f)(x + g)$$

$$(6) \quad (2x + 1)(x + 2) - 14$$

$$(2x - 1)(x + a) - b$$

$$(x + 6)(2x - c) + d$$

$$2(x - 3)(x + 2) + ex$$

$$(2x - f)(x + g)$$

Purposeful Practice

Prove it!

Prove that $(a + b)^2$ is not equal to $a^2 + b^2$

Show that $5a + 2a + 4 + 3$ is always a multiple of 7

$n - 1 + n + 1$ will be even. Why?

The sum of three consecutive numbers is 3 times the middle number

(hint - call your first number a , your second number $a + 1$)

The sum of two consecutive odd numbers is always even

For any three consecutive numbers, the difference between the square of the middle number and the product of the largest and smallest is always 1

Show that $5(2x + 5) + 11(x - 2) = 3(7x + 1)$

Purposeful Practice

from exercises in algebra part three by HF Browne (1936)

solve the equations

$$\frac{2}{x} + \frac{3}{y} = 4$$

$$\frac{5}{x} + \frac{6}{y} = 7$$

find two numbers such that
the sum of their squares is
five times their sum

and

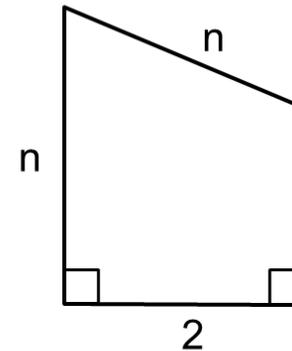
the difference of their squares
is nine times their difference

the perimeter of a
right-angled triangle is 40 cm
and the area is 60 cm^2

find the lengths of the sides

the area of the
right trapezium
is 4

find n



Fluency Practice

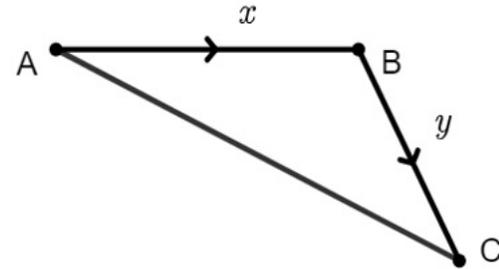
Number and Algebra Proof Revision			
(a)	(b)	(c)	(d)
Show that $8\frac{1}{2} - 3\frac{2}{3} = 4\frac{5}{6}$	Show that $\sqrt{80}$ can be written in the form $k\sqrt{5}$ where k is an integer to be found	Show that $0.\dot{7}\dot{5} = \frac{25}{33}$	Show that $3\frac{5}{8} \div 1\frac{5}{6} = 1\frac{43}{44}$
(e)	(f)	(g)	(h)
Show that $0.3\dot{1} = \frac{14}{45}$	Show that $0.4\dot{2}\dot{7} = \frac{47}{110}$	Show that $(7 - 5\sqrt{3})^2 = a + b\sqrt{3}$ where a and b are integers to be found	Show that the product of an even number and an odd number is always even.
(i)		(j)	
Show that the sum of three consecutive odd numbers is always a multiple of 3		Show that $\frac{3\sqrt{12}}{2-\sqrt{3}}$ can be written in the form $c + d\sqrt{3}$, where c and d are integers to be found.	
(k)			
Show that $(3n + 4)(n - 3) + n(n - 3)$ is a multiple of 4 for all integer values of n			

7 Advanced Vectors

Fluency Practice

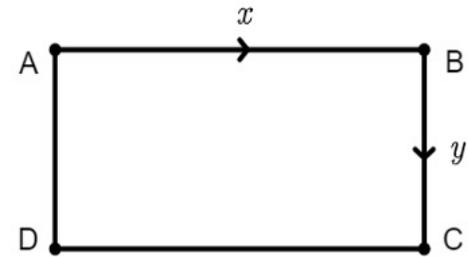
1. ABC is a triangle. Find, in terms of x and y :

- (a) \overrightarrow{BA} (b) \overrightarrow{CB}
 (c) \overrightarrow{AC} (d) \overrightarrow{CA}



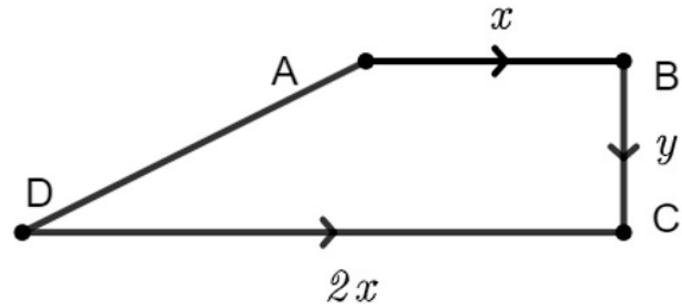
2. ABCD is a rectangle. Find, in terms of x and y :

- (a) \overrightarrow{DA} (b) \overrightarrow{AC}
 (c) \overrightarrow{CA} (d) \overrightarrow{BD}



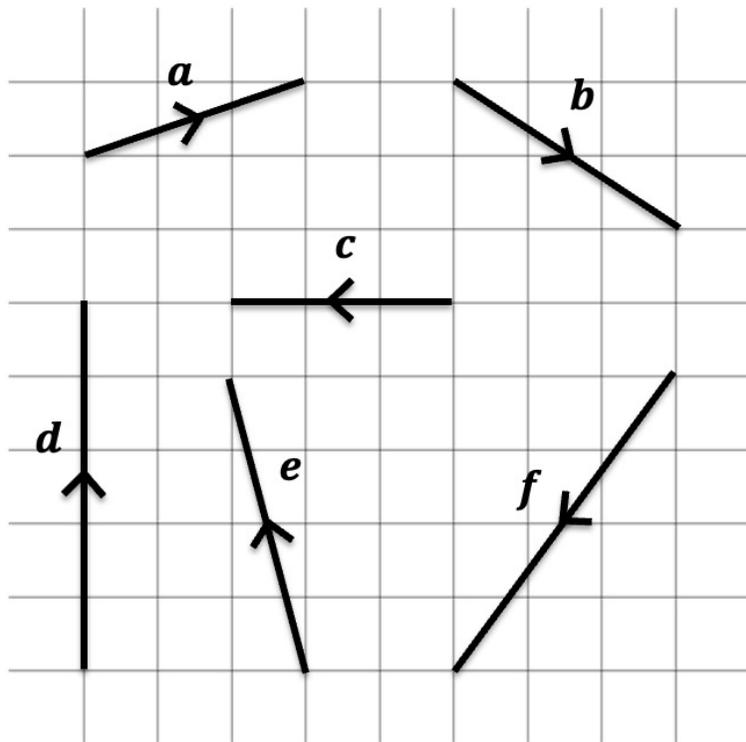
3. ABCD is a trapezium. Find, in terms of x and y :

- (a) \overrightarrow{AC} (b) \overrightarrow{DB}
 (c) \overrightarrow{AD} (d) \overrightarrow{DA}



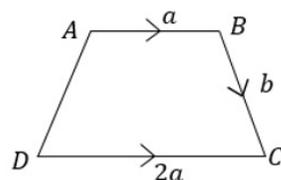
Fluency Practice

1 State the value of each vector.



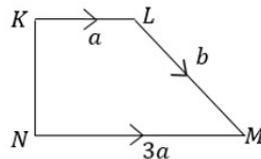
$a =$	$b =$	$c =$
$d =$	$e =$	$f =$

2



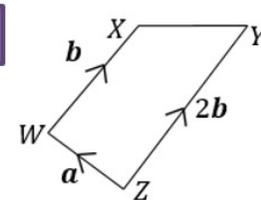
- a. $\overrightarrow{BA} =$
- b. $\overrightarrow{AC} =$
- c. $\overrightarrow{DB} =$
- d. $\overrightarrow{AD} =$

3



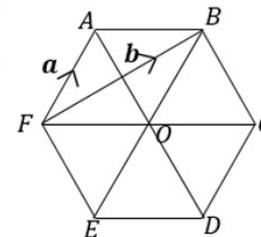
- a. $\overrightarrow{MK} =$
- b. $\overrightarrow{NL} =$
- c. $\overrightarrow{NK} =$
- d. $\overrightarrow{KN} =$

4



- a. $\overrightarrow{ZX} =$
- b. $\overrightarrow{YW} =$
- c. $\overrightarrow{XY} =$
- d. $\overrightarrow{XZ} =$

5



- a. $\overrightarrow{AB} =$
- b. $\overrightarrow{FO} =$
- c. $\overrightarrow{AO} =$
- d. $\overrightarrow{FD} =$

Purposeful Practice

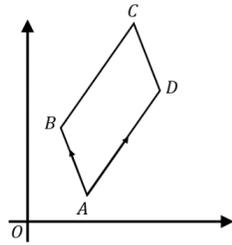
Vectors and Coordinates

(a)

$ABCD$ is a parallelogram.

$$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \text{ and } \overrightarrow{AD} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Given that the coordinates of A are $(3, 1)$, find the coordinates of points B , C and D .

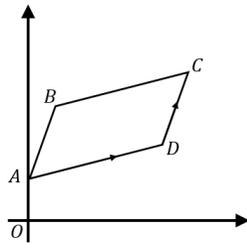


(b)

$ABCD$ is a parallelogram.

$$\overrightarrow{AD} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{DC} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

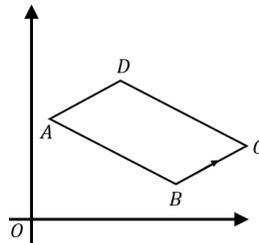
Given that the coordinates of A are $(0, 1)$, find the coordinates of points B , C and D .



(c)

$ABCD$ is a parallelogram. $\overrightarrow{BC} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

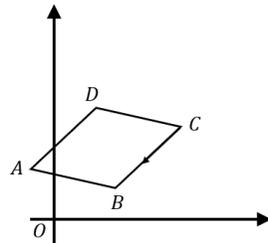
The coordinates of A are $(2, 7)$ and of B are $(8, 3)$. Find the coordinates of C and D , and the vector \overrightarrow{DC} .



(d)

$ABCD$ is a rhombus. $\overrightarrow{CB} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$

The coordinates of A are $(-1, 4)$ and of B are $(3, 1)$. Find the coordinates of C and D , and the vector \overrightarrow{DC} .

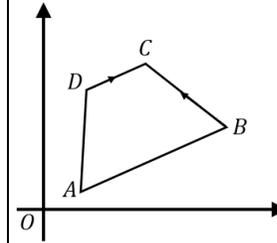


(e)

$ABCD$ is a trapezium. $\overrightarrow{AB} = 2\overrightarrow{DC}$.

$$\overrightarrow{DC} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{BC} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

The coordinates of D are $(2, 8)$. Find the coordinates of A , and the vector \overrightarrow{AD} .

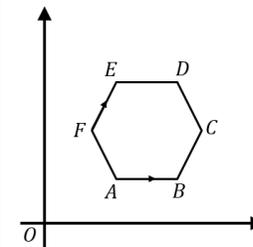


(f)

$ABCDEF$ is a regular hexagon.

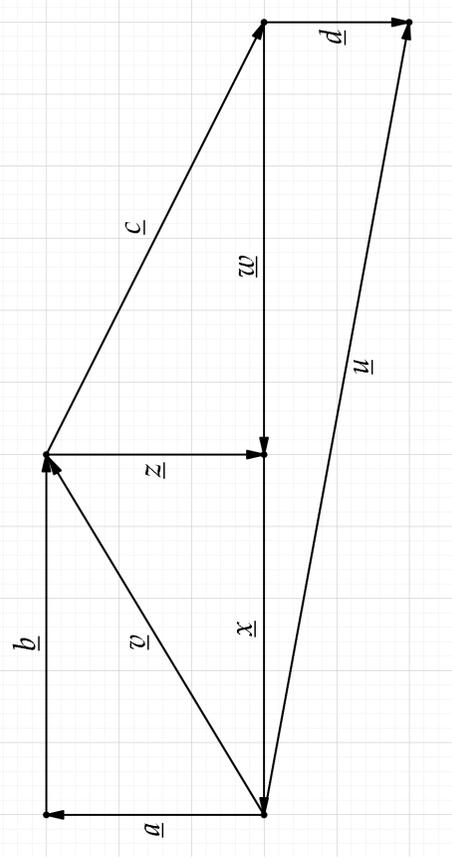
$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \text{ and } \overrightarrow{FE} = \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$$

$\overrightarrow{FC} = 2\overrightarrow{AB}$. The coordinates of A are $(5, 2)$. Find the coordinates of B , C and D .



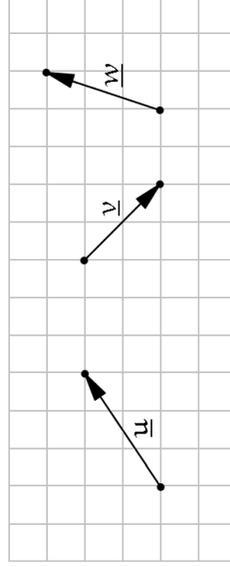
Fluency Practice

1. Based on the picture, determine whether each of the given statements is true or false.



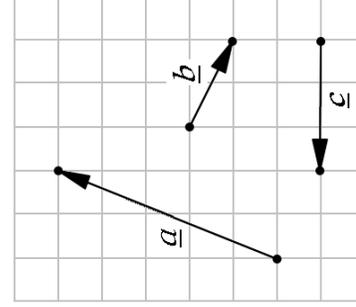
- | | | |
|--|--|---|
| a) $\underline{v} = \underline{a} + \underline{b}$ | f) $\underline{w} + \underline{x} + \underline{u} = \underline{d}$ | k) $\underline{x} + \underline{w} + \underline{v} = -\underline{c}$ |
| b) $\underline{v} = \underline{z} + \underline{x}$ | g) $\underline{d} + \underline{u} = \underline{x} + \underline{w}$ | l) $\underline{v} + \underline{c} + \underline{d} = -\underline{u}$ |
| c) $\underline{v} = \underline{x} - \underline{z}$ | h) $\underline{w} - \underline{c} = \underline{x} - \underline{v}$ | m) $\underline{a} + \underline{z} = \underline{0}$ |
| d) $\underline{c} = \underline{d} - \underline{u} + \underline{v}$ | i) $\underline{a} + \underline{b} + \underline{z} + \underline{x} = \underline{0}$ | n) $\underline{a} - \underline{b} = -\underline{v}$ |
| e) $\underline{a} = \underline{z}$ | j) $\underline{a} + \underline{b} + \underline{c} + \underline{w} + \underline{x} = \underline{0}$ | o) $\underline{z} - \underline{w} = \underline{c}$ |

2. Given the vectors \underline{u} , \underline{v} and \underline{w} , draw each of the following vectors.



- $\underline{u} + \underline{v}$
- $\underline{v} - \underline{w}$
- $2\underline{u} - \underline{v} + \underline{w}$

3. Given the vectors \underline{a} , \underline{b} and \underline{c} , draw each of the following vectors.



- $\underline{a} + \underline{b}$
- $3\underline{b}$
- $\underline{b} - \underline{a}$
- $\underline{a} + \underline{b} + \underline{c}$
- $\underline{a} - 2\underline{b} + \underline{c}$
- $-2\underline{c}$
- $-2\underline{a} + 3\underline{b} - \underline{c}$

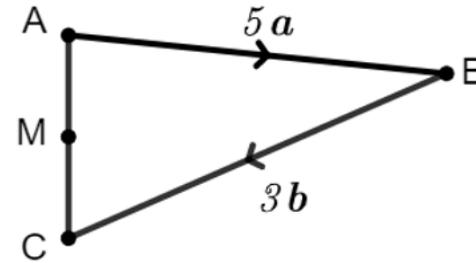
Purposeful Practice

4. Suppose that M is the midpoint of line segment AB . Let O be a point in the plane of AB . Let \underline{a} denote the vector pointing from O to A , let \underline{b} denote the vector pointing from O to B , and \underline{m} denote the vector pointing from O to M . Prove that $\underline{m} = \frac{\underline{a} + \underline{b}}{2}$. (Hint: Let \underline{x} be the vector pointing from A to M . Express first \underline{x} in terms of \underline{a} and \underline{b} . $\underline{m} = \underline{a} + \underline{x}$).

Fluency Practice

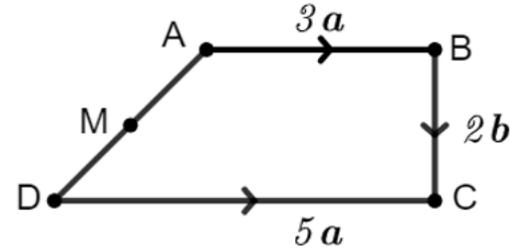
1. ABC is a triangle. M is the midpoint of AC. Find, in terms of a and b :

(a) \overrightarrow{AC} (b) \overrightarrow{CA}
 (c) \overrightarrow{AM} (d) \overrightarrow{BM}



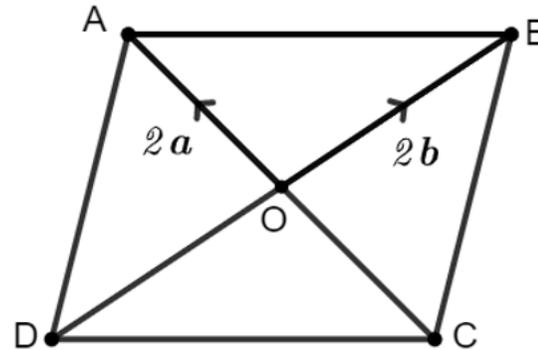
2. ABCD is a trapezium. M is the midpoint of AD. Find, in terms of a and b :

(a) \overrightarrow{DA} (b) \overrightarrow{DM}
 (c) \overrightarrow{AD} (d) \overrightarrow{MC}



3. ABCD is a parallelogram. Find, in terms of a and b :

(a) \overrightarrow{OC} (b) \overrightarrow{AC}
 (c) \overrightarrow{AB} (d) \overrightarrow{AD}



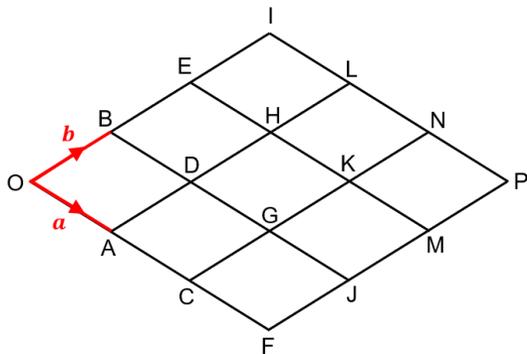
Fluency Practice

Defining Vectors

(a)

The diagram is made up of nine congruent rhombuses.

$$\vec{OA} = \mathbf{a} \text{ and } \vec{OB} = \mathbf{b}.$$



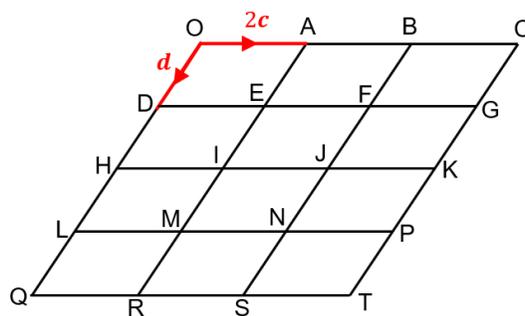
Define the following vectors in terms of \mathbf{a} and \mathbf{b} .

- | | |
|----------------|----------------|
| (a) \vec{OE} | (g) \vec{OH} |
| (b) \vec{OF} | (h) \vec{DP} |
| (c) \vec{GJ} | (i) \vec{IM} |
| (d) \vec{MP} | (j) \vec{MD} |
| (e) \vec{AO} | (k) \vec{CB} |
| (f) \vec{LD} | (l) \vec{NO} |

(b)

The diagram is made up of twelve congruent parallelograms.

$$\vec{OA} = 2\mathbf{c} \text{ and } \vec{OD} = \mathbf{d}.$$



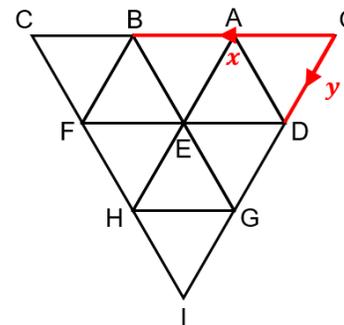
Define the following vectors in terms of \mathbf{c} and \mathbf{d} .

- | | |
|----------------|----------------|
| (a) \vec{EG} | (g) \vec{MJ} |
| (b) \vec{BS} | (h) \vec{QF} |
| (c) \vec{KJ} | (i) \vec{IL} |
| (d) \vec{RE} | (j) \vec{HB} |
| (e) \vec{OF} | (k) \vec{TE} |
| (f) \vec{JT} | (l) \vec{KD} |

(c)

The diagram is made up of nine congruent equilateral triangles.

$$\vec{OB} = \mathbf{x} \text{ and } \vec{OD} = \mathbf{y}.$$



Define the following vectors in terms of \mathbf{x} and \mathbf{y} .

- | | |
|----------------|----------------|
| (a) \vec{FD} | (g) \vec{FO} |
| (b) \vec{HA} | (h) \vec{CE} |
| (c) \vec{GH} | (i) \vec{HI} |
| (d) \vec{OC} | (j) \vec{BG} |
| (e) \vec{DB} | (k) \vec{HD} |
| (f) \vec{OE} | (l) \vec{IF} |

Purposeful Practice

Vectors and Midpoints

(a)

In the triangle OAB , $\overrightarrow{OA} = 2\mathbf{a}$ and $\overrightarrow{OB} = 4\mathbf{b}$. C is the midpoint of the line AB .

Express the following in terms of \mathbf{a} and \mathbf{b} :

(a) \overrightarrow{AB}

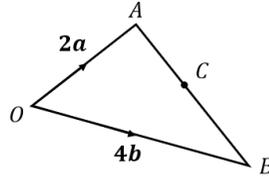
(b) \overrightarrow{BA}

(c) \overrightarrow{AC}

(d) \overrightarrow{BC}

(e) \overrightarrow{OC}

(f) \overrightarrow{CO}



(b)

$OABC$ is a trapezium, where $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{AB} = \mathbf{b}$. D is the midpoint of BC and $\overrightarrow{OC} = 2\overrightarrow{AB}$.

Express the following in terms of \mathbf{a} and \mathbf{b} :

(a) \overrightarrow{OC}

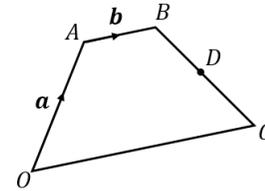
(b) \overrightarrow{CB}

(c) \overrightarrow{BC}

(d) \overrightarrow{BD}

(e) \overrightarrow{AD}

(f) \overrightarrow{DO}



(c)

In the parallelogram $OABC$, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. X is the midpoint of the line OB .

Express the following in terms of \mathbf{a} and \mathbf{c} :

(a) \overrightarrow{CB}

(b) \overrightarrow{BA}

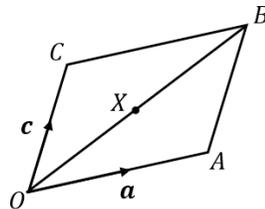
(c) \overrightarrow{OB}

(d) \overrightarrow{XB}

(e) \overrightarrow{XC}

(f) \overrightarrow{AX}

What do the answers to (e) and (f) tell us about the points C , X and A ?



(d)

$OABC$ is a quadrilateral. $\overrightarrow{OX} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{c}$ and $\overrightarrow{CB} = \mathbf{b}$. X is the midpoint of OA and Y is the midpoint of AB .

Express the following in terms of \mathbf{a} , \mathbf{b} and \mathbf{c} :

(a) \overrightarrow{OA}

(b) \overrightarrow{OB}

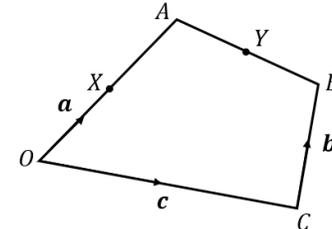
(c) \overrightarrow{CA}

(d) \overrightarrow{AB}

(e) \overrightarrow{AY}

(f) \overrightarrow{XY}

What do the answers to (b) and (f) tell us about vectors \overrightarrow{OB} and \overrightarrow{XY} ?



Fluency Practice

Vectors Around Shapes

$$\vec{WY} =$$

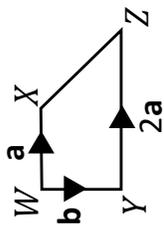
$$\vec{WZ} =$$

$$\vec{XZ} =$$

1) $\vec{WX} =$

$$\vec{YZ} =$$

$$\vec{XY} =$$



$$\vec{WV} =$$

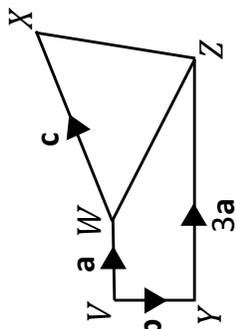
$$\vec{VZ} =$$

$$\vec{XZ} =$$

2) $\vec{VX} =$

$$\vec{XV} =$$

$$\vec{WZ} =$$



$$\vec{VY} =$$

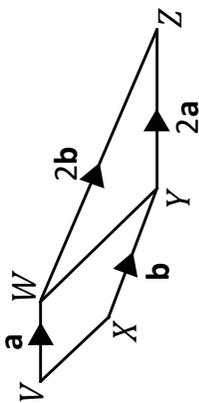
$$\vec{WY} =$$

$$\vec{XV} =$$

3) $\vec{VZ} =$

$$\vec{XZ} =$$

$$\vec{YW} =$$



$$\vec{VX} =$$

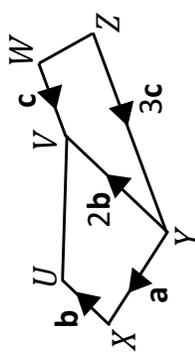
$$\vec{UY} =$$

$$\vec{UW} =$$

4) $\vec{YZ} =$

$$\vec{ZW} =$$

$$\vec{UZ} =$$



$$\vec{WT} =$$

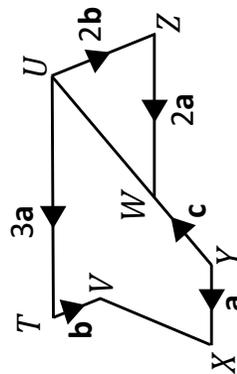
$$\vec{XZ} =$$

$$\vec{XV} =$$

5) $\vec{VZ} =$

$$\vec{ZX} =$$

$$\vec{YT} =$$



6) (V is the midpoint of UW)

$$\vec{WV} =$$

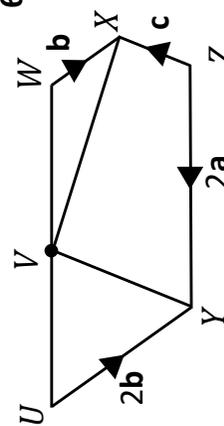
$$\vec{UV} =$$

$$\vec{YV} =$$

$$\vec{WU} =$$

$$\vec{UW} =$$

$$\vec{XV} =$$

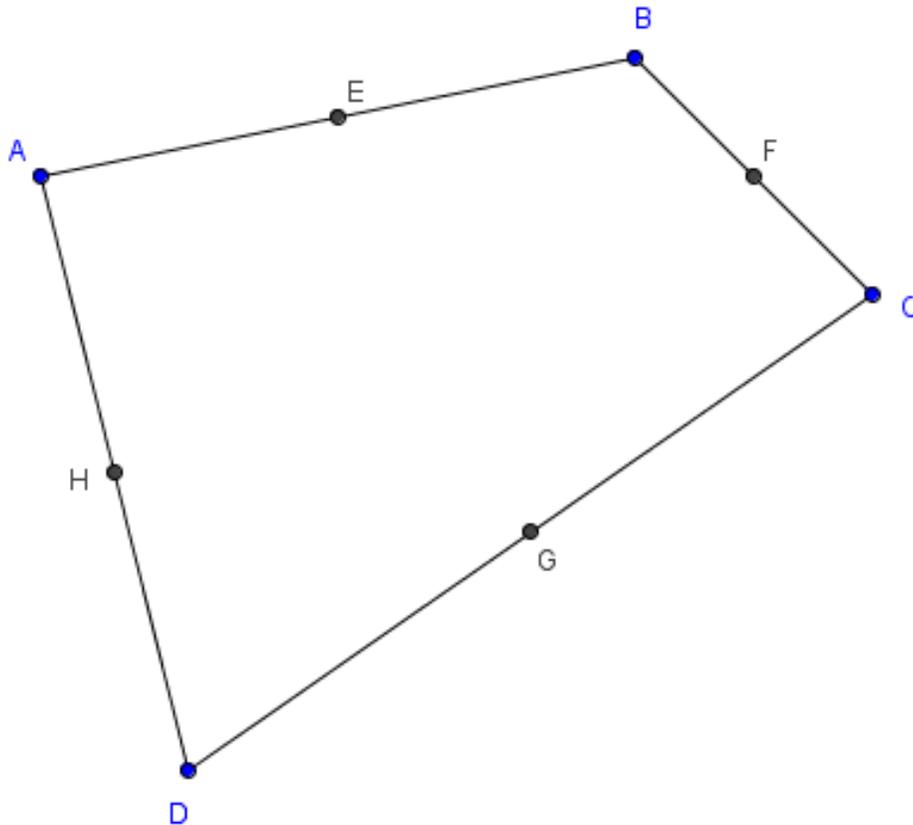


Purposeful Practice

The diagram shows the quadrilateral ABCD.

The point E is the midpoint of A and B. The point F is the midpoint of B and C.

The point G is the midpoint of C and D. The point H is the midpoint of D and A.



$$\overrightarrow{AB} = \mathbf{a} \quad \overrightarrow{BC} = \mathbf{b} \quad \overrightarrow{CD} = \mathbf{c}$$

1. Express in terms of \mathbf{a} , \mathbf{b} and/or \mathbf{c} :

$$\overrightarrow{DC} =$$

$$\overrightarrow{AD} =$$

2. Express in terms of \mathbf{a} , \mathbf{b} and/or \mathbf{c} :

$$\overrightarrow{HE} =$$

$$\overrightarrow{GF} =$$

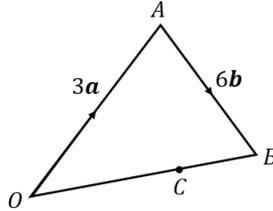
What do you notice?

Purposeful Practice

Vectors and Ratio

(a)

In the triangle OAB , $\overrightarrow{OA} = 3\mathbf{a}$ and $\overrightarrow{AB} = 6\mathbf{b}$. The point C divides the line OB in the ratio $2 : 1$.



Express the following in terms of \mathbf{a} and \mathbf{b} :

(a) \overrightarrow{OB}

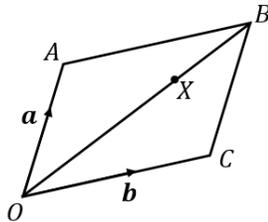
(b) \overrightarrow{OC}

(c) \overrightarrow{BC}

(d) \overrightarrow{AC}

(c)

In the parallelogram $OACB$, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{b}$. The point X divides the line OB in the ratio $3 : 2$.



Express the following in terms of \mathbf{a} and \mathbf{b} :

(a) \overrightarrow{AB}

(b) \overrightarrow{OB}

(c) \overrightarrow{OX}

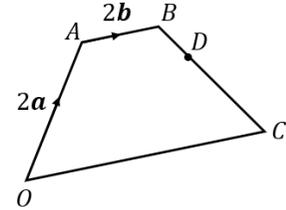
(d) \overrightarrow{BX}

(e) \overrightarrow{AX}

(f) \overrightarrow{XC}

(b)

$OABC$ is a trapezium, where $\overrightarrow{OA} = 2\mathbf{a}$ and $\overrightarrow{AB} = 2\mathbf{b}$. $\overrightarrow{OC} = 2\overrightarrow{AB}$ and D splits the line BC in the ratio $1 : 3$.



Express the following in terms of \mathbf{a} and \mathbf{b} :

(a) \overrightarrow{BO}

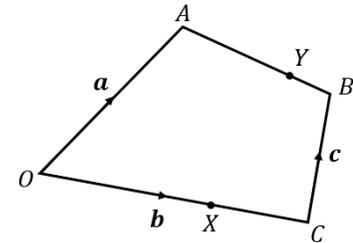
(b) \overrightarrow{BC}

(c) \overrightarrow{BD}

(d) \overrightarrow{DO}

(d)

$OABC$ is a quadrilateral. $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{b}$ and $\overrightarrow{CB} = \mathbf{c}$. The point X divides the line OC in the ratio $2 : 1$. The point Y divides the line AB in the ratio $3 : 1$.



Express the following in terms of \mathbf{a} , \mathbf{b} and \mathbf{c} :

(a) \overrightarrow{OX}

(b) \overrightarrow{XC}

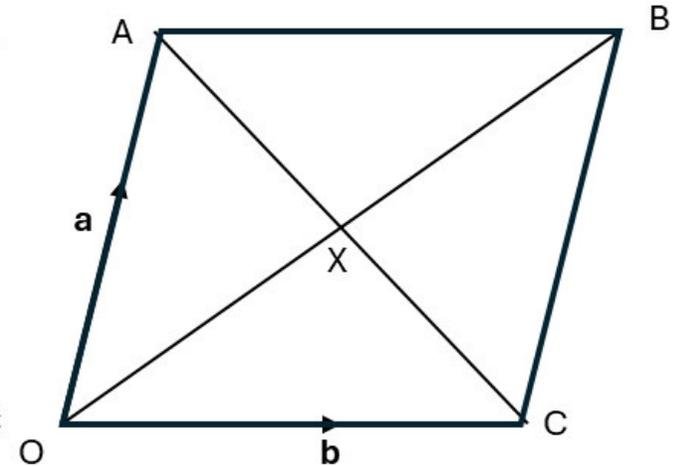
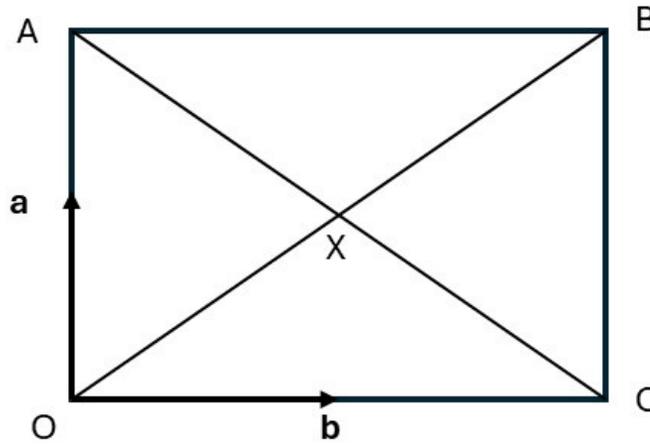
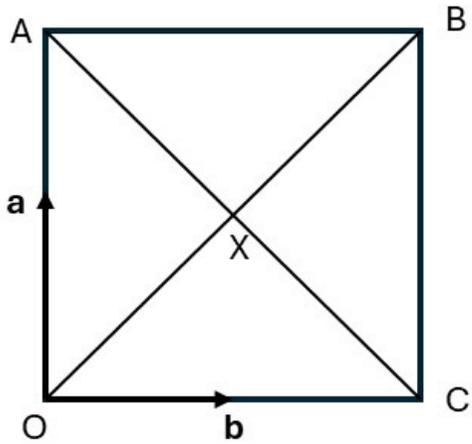
(c) \overrightarrow{AB}

(d) \overrightarrow{AY}

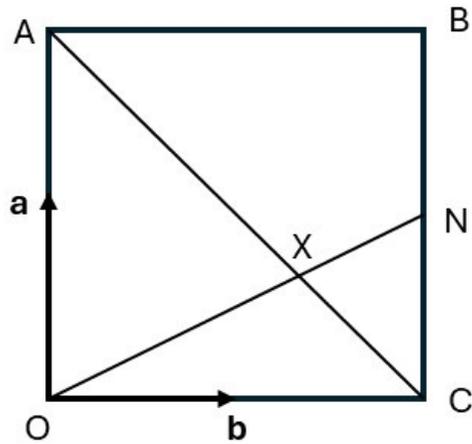
(e) \overrightarrow{AX}

(f) \overrightarrow{XY}

Purposeful Practice

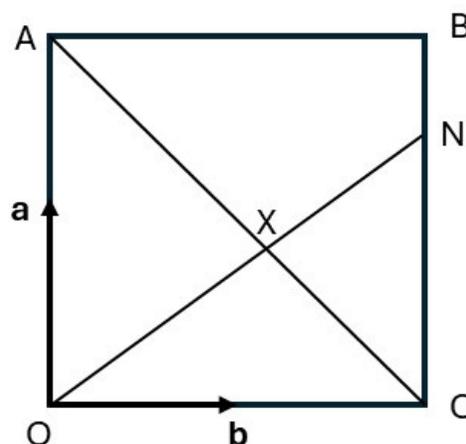


What is the ratio of $OX:XB$ and $AX:AC$

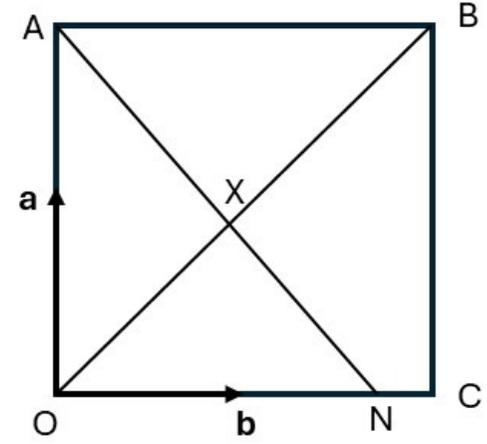


N is the midpoint of BC

What is the ratio of $OX:OX$ and $AX:AC$



Ratio of $BN:NC$ is 1:3



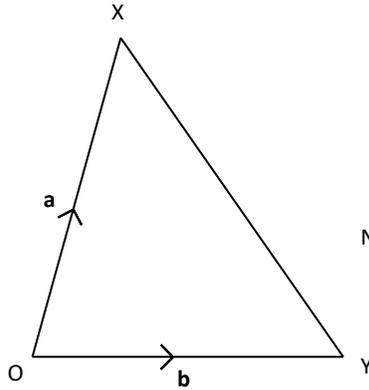
Ratio of $ON:OC$ is 4:5

What fraction of AN is AX?

Purposeful Practice

1)

M is the midpoint of line XY.



Vectors Around Shapes

$$\overrightarrow{YX} =$$

$$\overrightarrow{YM} =$$

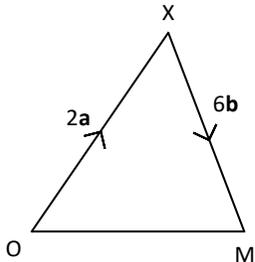
$$\overrightarrow{OM} =$$

N is the midpoint of line OM.

$$\overrightarrow{ON} =$$

2)

M divides the line OY in the ratio 2:3



$$\overrightarrow{OM} =$$

$$\overrightarrow{OY} =$$

$$\overrightarrow{MY} =$$

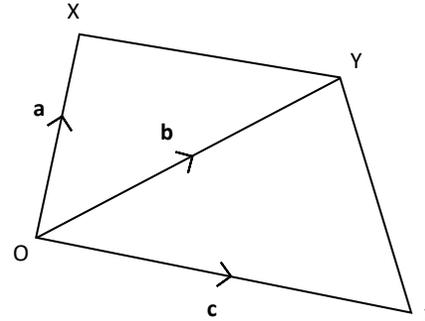
$$\overrightarrow{YM} =$$

$$\overrightarrow{YX} =$$

3)

M is the midpoint of line XY.

N divides the line ZY in the ratio 1:2



$$\overrightarrow{XM} =$$

$$\overrightarrow{ZN} =$$

Express using a, b & c.

$$\overrightarrow{ZM} =$$

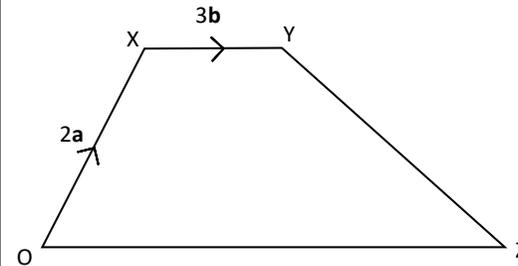
$$\overrightarrow{MN} =$$

4)

OZ = 2XY

M divides the line XZ in the ratio 1:2

N divides the line YZ in the ratio 1:2



$$\overrightarrow{XZ} =$$

$$\overrightarrow{XM} =$$

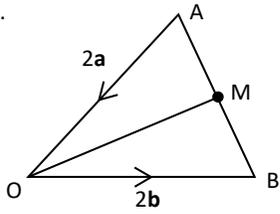
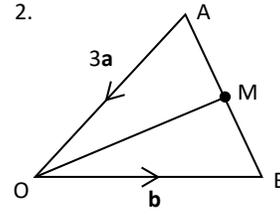
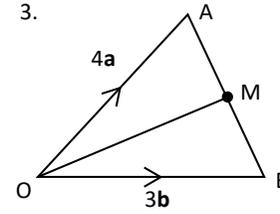
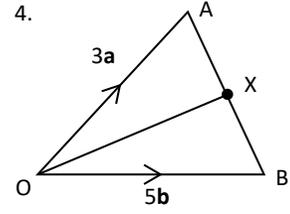
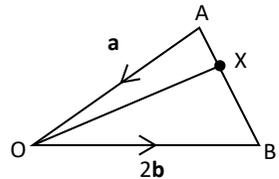
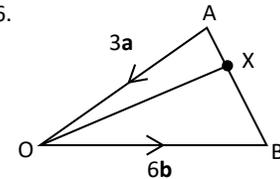
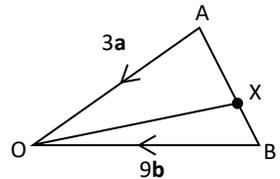
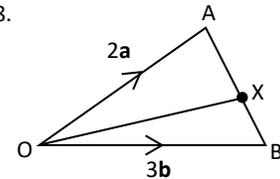
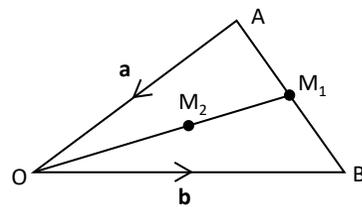
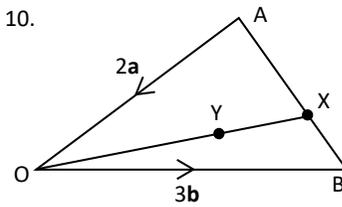
$$\overrightarrow{YZ} =$$

$$\overrightarrow{YN} =$$

$$\overrightarrow{MN} =$$

What statements can you make about the magnitude & direction of lines XY & MN?

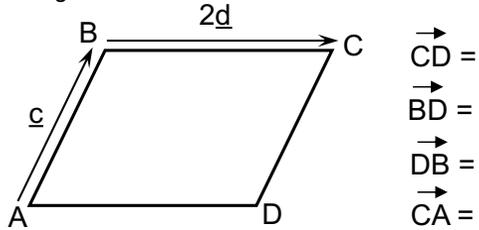
Purposeful Practice

Vector Proofs			
<p>1.</p>  <p>Prove $\overrightarrow{AM} = \mathbf{a} + \mathbf{b}$</p>	<p>2.</p>  <p>Prove $\overrightarrow{AM} = \frac{3}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$</p>	<p>M is the midpoint of the line.</p> <p>3.</p>  <p>Prove $\overrightarrow{AM} = \frac{3}{2}\mathbf{b} - 2\mathbf{a}$</p>	<p>4.</p>  <p>$AX:AB = 1:2$ Prove $\overrightarrow{AX} = 2.5\mathbf{b} - 1.5\mathbf{a}$</p>
<p>5.</p>  <p>$AX:AB = 1:3$ Prove $\overrightarrow{AX} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$</p>	<p>6.</p>  <p>$AX:XB = 1:2$ Prove $\overrightarrow{AX} = \mathbf{a} + 2\mathbf{b}$</p>	<p>7.</p>  <p>$AX:XB = 2:1$ Prove $\overrightarrow{OX} = -\mathbf{a} - 6\mathbf{b}$</p>	<p>8.</p>  <p>$AX:XB = 3:2$ Prove $\overrightarrow{AX} = 1.8\mathbf{b} - 1.2\mathbf{a}$</p>
<p>9.</p>  <p>Prove $\overrightarrow{AM_2} = \frac{1}{4}\mathbf{b} - \frac{1}{4}\mathbf{a}$</p>	<p>10.</p>  <p>$AX:XB = 2:1$ $OY:YX = 2:1$ Prove $\overrightarrow{OY} = \frac{4}{3}\mathbf{b} - \frac{4}{9}\mathbf{a}$</p>		

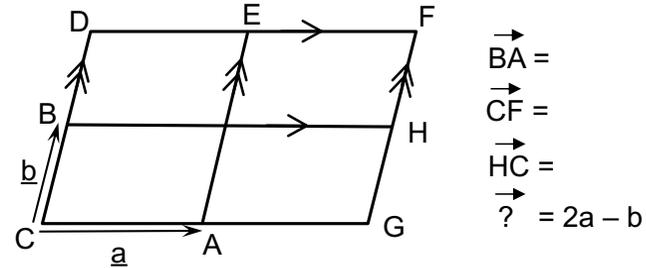
Purposeful Practice

vector questions (i)

a parallelogram

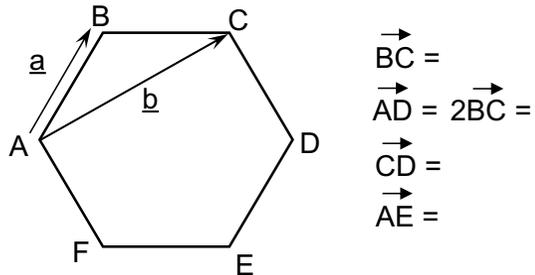


a parallelogram



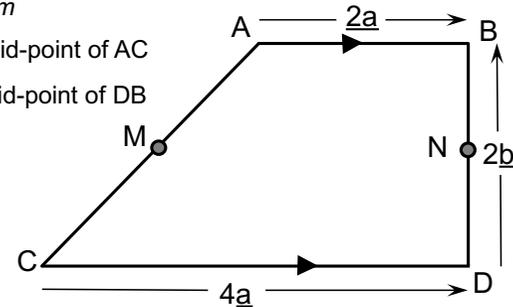
A is the mid-point of CG B is the mid-point of CD

a regular hexagon



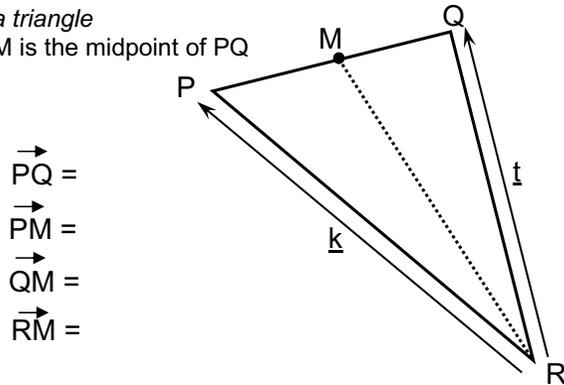
a trapezium

M is the mid-point of AC
N is the mid-point of DB



a triangle

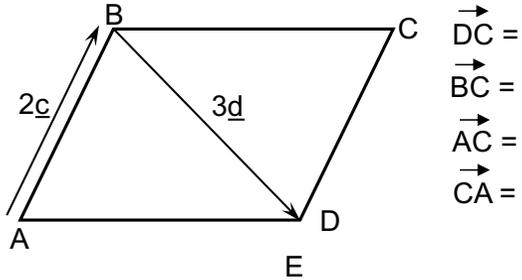
M is the midpoint of PQ



Purposeful Practice

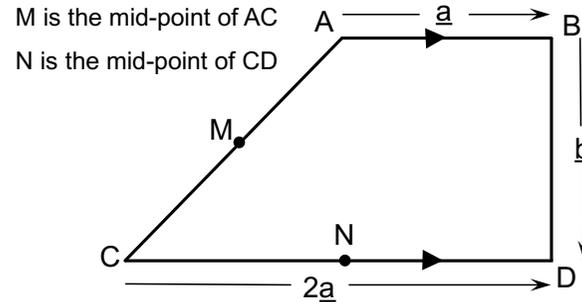
vector questions (ii)

a parallelogram



$$\begin{aligned} \vec{DC} &= \\ \vec{BC} &= \\ \vec{AC} &= \\ \vec{CA} &= \end{aligned}$$

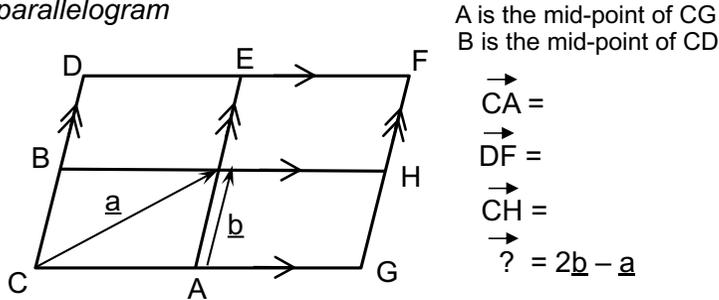
a trapezium



M is the mid-point of AC
 N is the mid-point of CD

$$\begin{aligned} \vec{AD} &= & \vec{CM} &= \\ \vec{CA} &= & \vec{MN} &= \end{aligned}$$

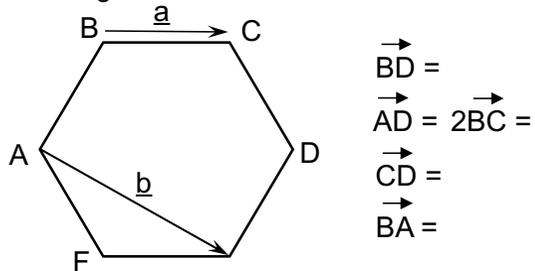
a parallelogram



A is the mid-point of CG
 B is the mid-point of CD

$$\begin{aligned} \vec{CA} &= \\ \vec{DF} &= \\ \vec{CH} &= \\ ? &= 2\mathbf{b} - \mathbf{a} \end{aligned}$$

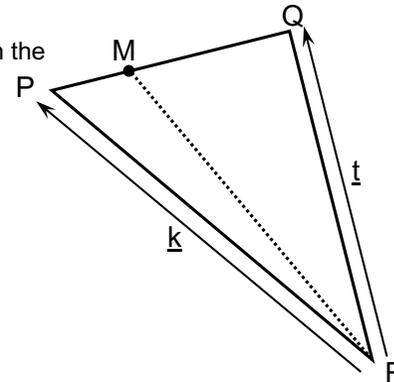
a regular hexagon



$$\begin{aligned} \vec{BD} &= \\ \vec{AD} &= 2\vec{BC} = \\ \vec{CD} &= \\ \vec{BA} &= \end{aligned}$$

a triangle

M divides PQ in the ratio $1 : 2$

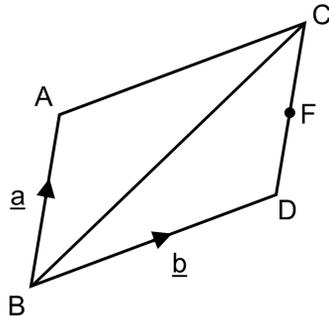


$$\begin{aligned} \vec{PQ} &= \\ \vec{PM} &= \\ \vec{QM} &= \\ \vec{RM} &= \end{aligned}$$

Purposeful Practice

vector questions (iii)

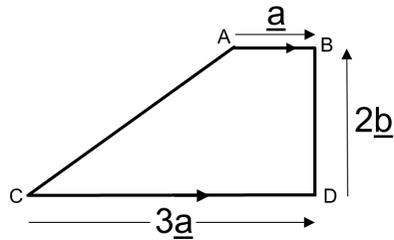
ACDB is a parallelogram



F is the midpoint of CD.

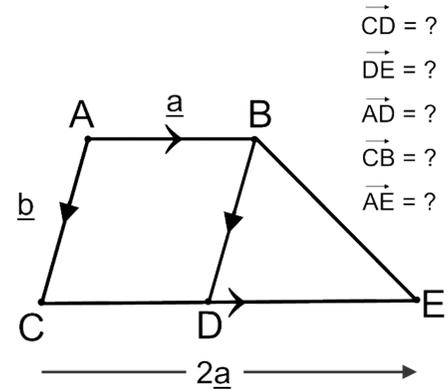
work out the vectors:

- $\vec{DC} = ?$
- $\vec{DF} = ?$
- $\vec{AD} = ?$
- $\vec{CB} = ?$
- $\vec{BF} = ?$

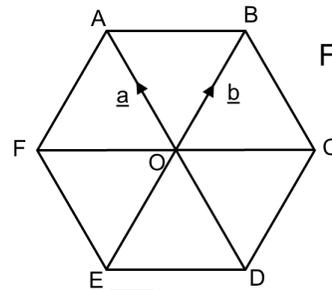


find, in terms of a and b, the vectors:

- 1) \vec{CA}
- 2) \vec{CB}
- 3) \vec{DA}



- $\vec{CD} = ?$
- $\vec{DE} = ?$
- $\vec{AD} = ?$
- $\vec{CB} = ?$
- $\vec{AE} = ?$



FEDCBA is a regular hexagon

- $OA = \underline{a}$
- $OB = \underline{b}$

write down, in terms of **a** and **b**, the vectors

- (i) \vec{AB}
- (ii) \vec{FC}

what fact can you deduce about \vec{AB} and \vec{FC} ?

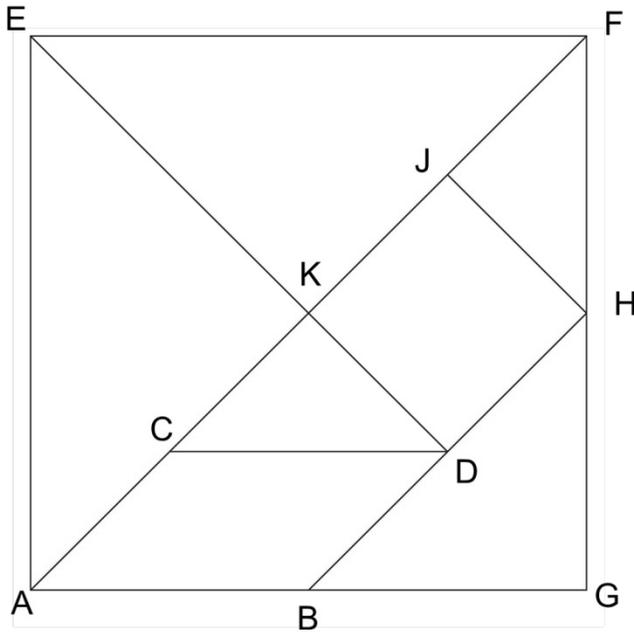
Purposeful Practice

vector questions (iv)

a tangram puzzle

$$AB = \underline{b}$$

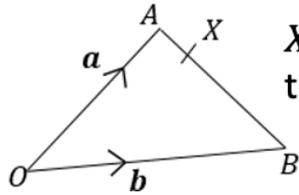
$$AC = \underline{c}$$



- $\vec{AF} =$
- $\vec{BH} =$
- $\vec{DK} =$
- $\vec{GH} =$
- $\vec{JC} =$
- $\vec{BH} =$
- $\vec{HC} =$
- $\vec{DE} =$

Purposeful Practice

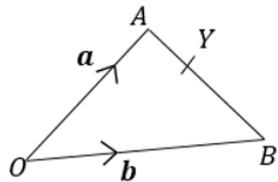
1



X is a point such that $AX:XB = 1:4$

- a. $\overrightarrow{AB} =$
- b. $\overrightarrow{AX} =$
- c. $\overrightarrow{OX} =$
- d. $\overrightarrow{BX} =$

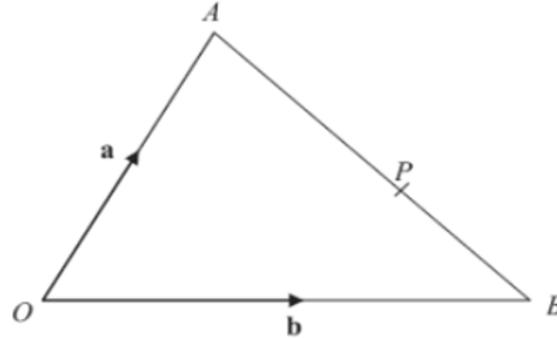
2



Y is a point such that $YB = 2AY$

- a. $\overrightarrow{AY} =$
- b. $\overrightarrow{OY} =$
- c. $\overrightarrow{YO} =$

3



[June 2009 2H Q23]

a) Find \overrightarrow{AB} in terms of \mathbf{a} and \mathbf{b} .

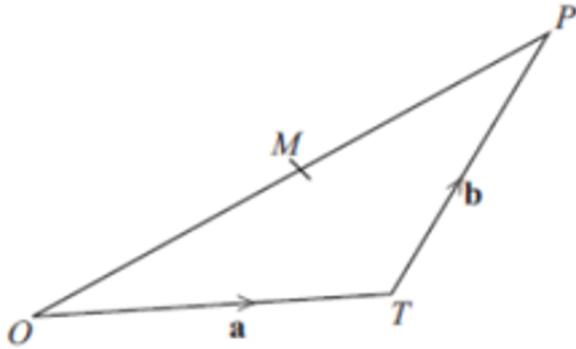
b) P is on AB such that $AP:PB = 3:2$.

Show that $\overrightarrow{OP} = \frac{1}{5}(2\mathbf{a} + 3\mathbf{b})$

?

Purposeful Practice

- 4** [Nov 2010 1H Q27] M is the midpoint of OP .



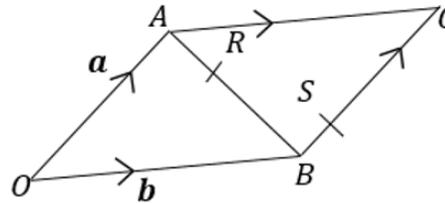
- a) Express \overrightarrow{OM} in terms of \mathbf{a} and \mathbf{b} .

?

- b) Express \overrightarrow{TM} in terms of \mathbf{a} and \mathbf{b} giving your answer in its simplest form.

?

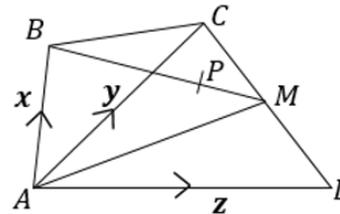
5



$OACB$ is a parallelogram. R is a point such that $AR:RC = 2:3$. S is a point such that $BS:SC = 1:3$.

- a. $\overrightarrow{OR} =$?
- b. $\overrightarrow{BS} =$?
- c. $\overrightarrow{OS} =$?
- d. $\overrightarrow{RS} =$?

6

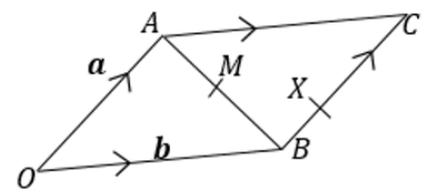


M is the midpoint of CD , $BP:PM = 2:1$

- a. $\overrightarrow{DC} =$?
- b. $\overrightarrow{DM} =$?
- c. $\overrightarrow{AM} =$?
- d. $\overrightarrow{BM} =$?
- e. $\overrightarrow{BP} =$?
- f. $\overrightarrow{AP} =$?

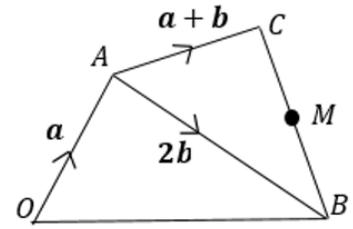
Purposeful Practice

7



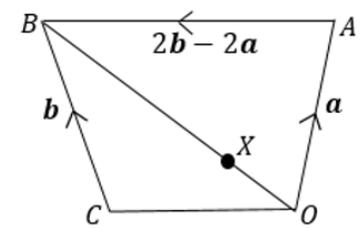
- a. $\overrightarrow{AB} =$
- b. $\overrightarrow{BC} =$
- c. $\overrightarrow{MB} =$
- d. $\overrightarrow{MX} =$
- e. $\overrightarrow{XA} =$
- f. $\overrightarrow{CM} =$
- g. $\overrightarrow{XO} =$

8



- a. $\overrightarrow{OB} =$
- b. $\overrightarrow{BC} =$
- c. $\overrightarrow{AM} =$
- d. $\overrightarrow{OM} =$

9



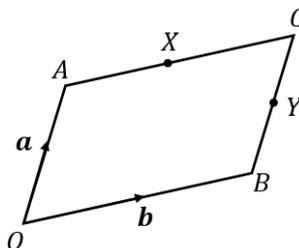
- a. $\overrightarrow{OC} =$
- b. $\overrightarrow{XC} =$
- c. $\overrightarrow{AX} =$

Purposeful Practice

Vector Proof – Parallel Lines

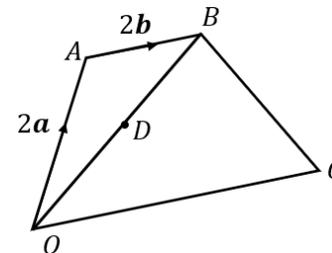
(a)

$OACB$ is a parallelogram. $\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$. X is the midpoint of AC and Y is the midpoint of BC . Show that XY and AB are parallel.



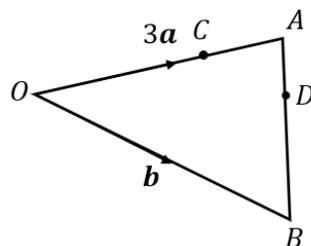
(b)

$OACB$ is a trapezium. $\vec{OA} = 2\mathbf{a}$ and $\vec{AB} = 2\mathbf{b}$. $\vec{OC} = 2\vec{AB}$ and D is the midpoint of OB . Show that AD is parallel to BC .



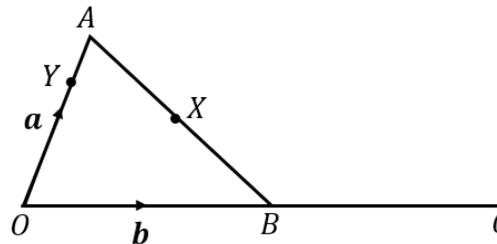
(c)

In the triangle OAB , $\vec{OB} = \mathbf{b}$ and $\vec{OA} = 3\mathbf{a}$. The point C divides the line OA in the ratio $2 : 1$ and the point D divides the line AB in the ratio $1 : 2$. Show that CD is parallel to OB .



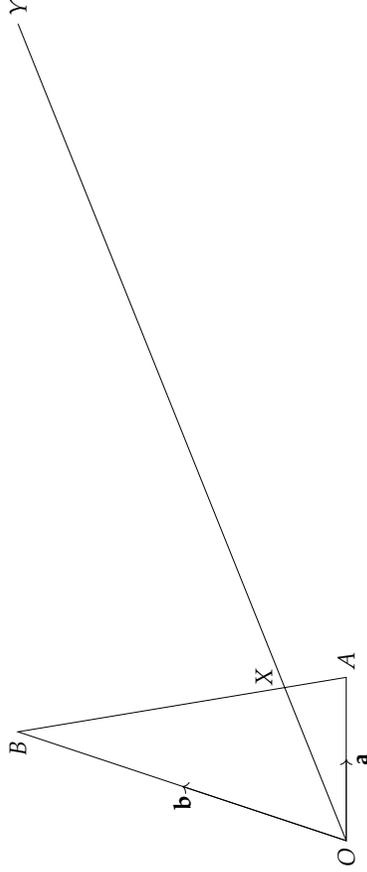
(d)

In the triangle OAB , $\vec{OB} = \mathbf{b}$ and $\vec{OA} = \mathbf{a}$. Point B is the midpoint of the line OC and X is the midpoint of AB . The point Y divides the line OA in the ratio $3 : 1$. Show that YX is parallel to AC .



Purposeful Practice

1.

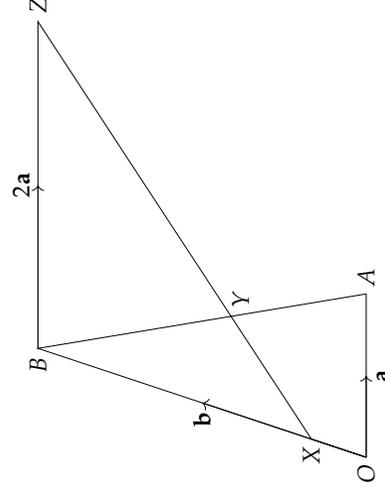


In the diagram,

- $\vec{OA} = \mathbf{a}$
- $\vec{OB} = \mathbf{b}$
- $AX : XB = 1 : 4$
- $\vec{OY} = 5 \times \vec{OX}$

Show that \vec{OA} and \vec{BY} are parallel.

2.



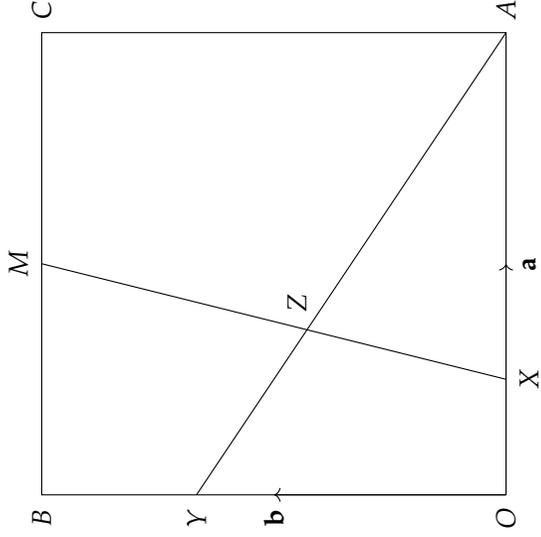
In the diagram,

- $\vec{OA} = \mathbf{a}$
- $\vec{OB} = \mathbf{b}$
- $\vec{BY} = 2\mathbf{b}$
- $OX : XB = 1 : 5$

Find the ratio $BY : BA$.

Purposeful Practice

3.

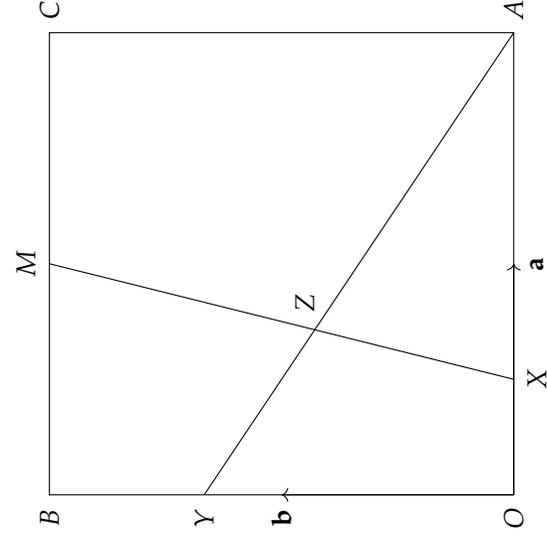


In the diagram,

- $ABCD$ is a square.
- $\vec{OA} = \mathbf{a}$
- $\vec{OB} = \mathbf{b}$
- M is the midpoint of BC
- $OY : YB = 2 : 1$
- $OX : XA = 1 : 3$

Find the vector \vec{OZ} .

4.



In the diagram,

- $ABCD$ is a square.

Purposeful Practice

- $\vec{OA} = \mathbf{a}$
- $\vec{OB} = \mathbf{b}$
- M is the midpoint of BC
- $\vec{OX} = m \times \vec{OA}$
- $\vec{OY} = n \times \vec{OB}$

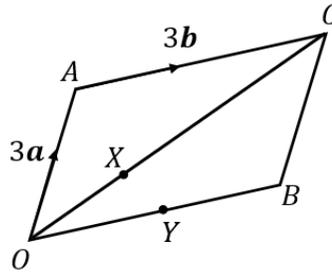
Find the vector \vec{OZ} , giving your answer in terms of m and n .

Purposeful Practice

Vector Proof – Collinear Points

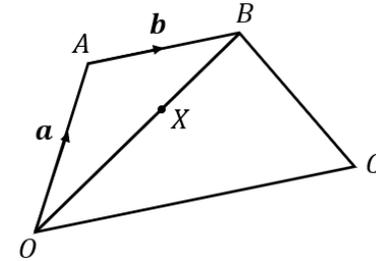
(a)

$OACB$ is a parallelogram. $\vec{OA} = 3\mathbf{a}$ and $\vec{AC} = 3\mathbf{b}$. Y is the midpoint of OB and X divides the line OC in the ratio 1 : 2. Show that the points A, X and Y are collinear.



(b)

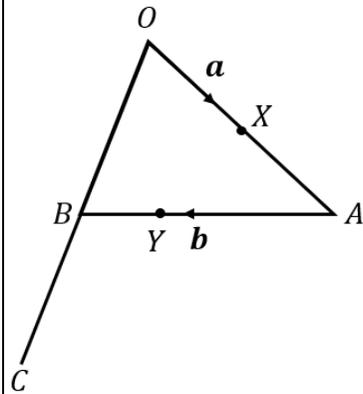
$OACB$ is a trapezium. $\vec{OA} = \mathbf{a}$ and $\vec{AB} = \mathbf{b}$. $\vec{OC} = 3\vec{AB}$ and X divides the line OB in the ratio 3 : 1. Show that the points A, X and C are collinear.



(c)

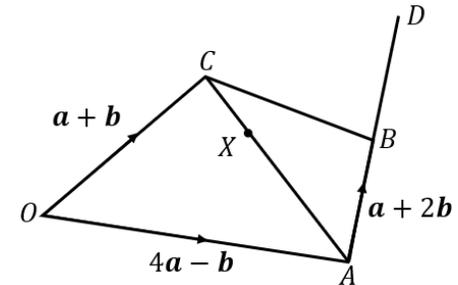
In the triangle OAB , $\vec{OX} = \mathbf{a}$ and $\vec{AB} = \mathbf{b}$. X is the midpoint of OA and the point Y divides the line AB in the ratio 2 : 1.

$\vec{OB} = \vec{BC}$. Show that the points X, Y and C are collinear.



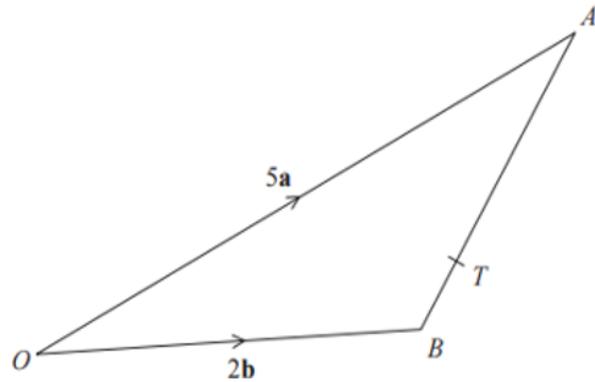
(d)

$\vec{OA} = 4\mathbf{a} - \mathbf{b}$, $\vec{AB} = \mathbf{a} + 2\mathbf{b}$ and $\vec{OC} = \mathbf{a} + \mathbf{b}$. $\vec{AB} = \vec{BD}$. The point X divides the line AC in the ratio 6 : 1. Show that O, X and D are collinear.

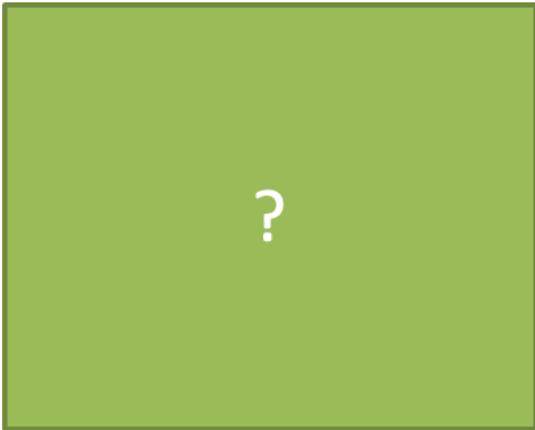


Purposeful Practice

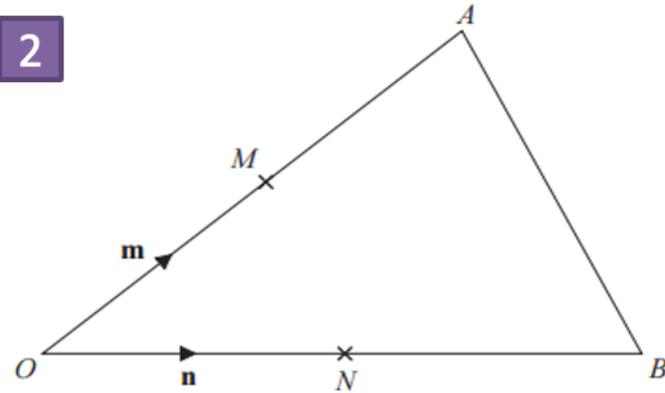
1



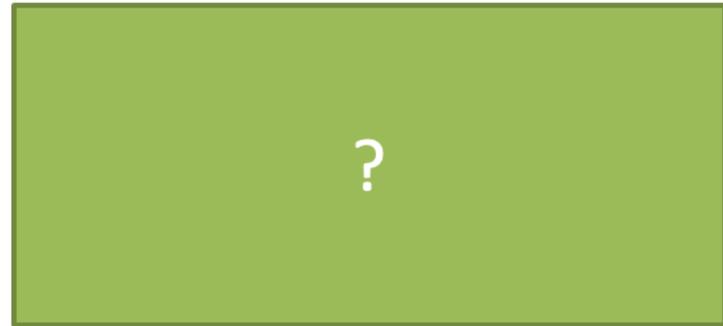
T is the point on AB such that $AT:TB = 5:1$. Show that OT is parallel to the vector $\mathbf{a} + 2\mathbf{b}$.



2

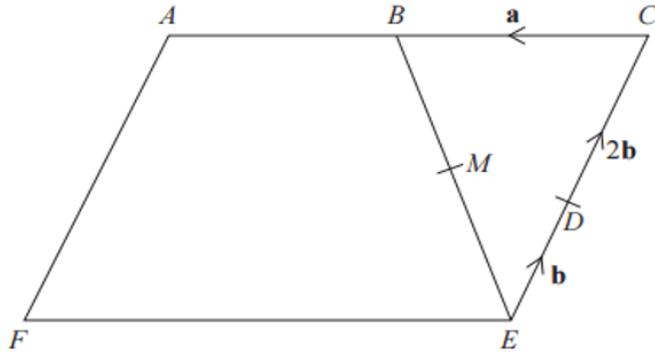


M is the midpoint of OA . N is the midpoint of OB . Prove that AB is parallel to MN .



Purposeful Practice

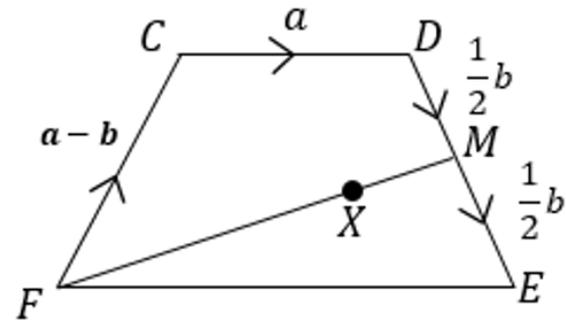
3



$ACEF$ is a parallelogram. B is the midpoint of AC . M is the midpoint of BE . Show that AMD is a straight line.



4



$\vec{CD} = \mathbf{a}$, $\vec{DE} = \mathbf{b}$ and $\vec{FC} = \mathbf{a} - \mathbf{b}$

i) Express \vec{CE} in terms of \mathbf{a} and \mathbf{b} .



ii) Prove that \vec{FE} is parallel to \vec{CD} .

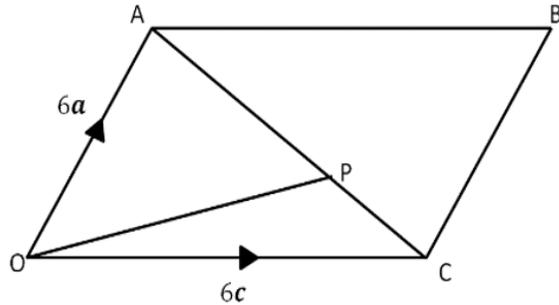


iii) X is the point on FM such that $FX:XM = 4:1$. Prove that C , X and E lie on the same straight line.



Purposeful Practice

5



$OABC$ is a parallelogram. P is the point on AC such that $AP = \frac{2}{3}AC$.

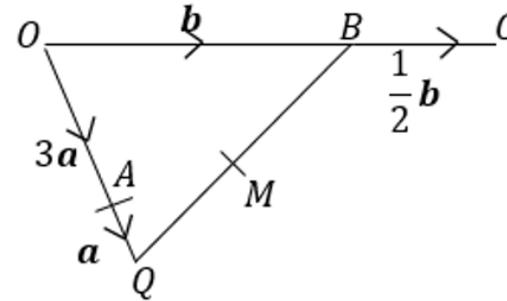
i) Find the vector \overrightarrow{OP} . Give your answer in terms of \mathbf{a} and \mathbf{c} .



ii) Given that the midpoint of CB is M , prove that OPM is a straight line.



6



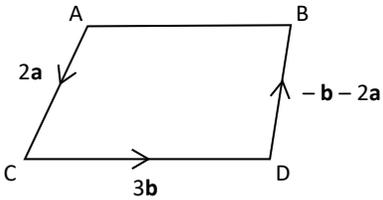
$\overrightarrow{OA} = 3\mathbf{a}$ and $\overrightarrow{AQ} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{BC} = \frac{1}{2}\mathbf{b}$. M is the midpoint of QB . Prove that AMC is a straight line.



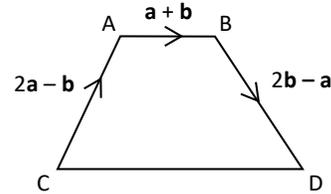
Purposeful Practice

Vectors: Proving Parallel & Straight Lines

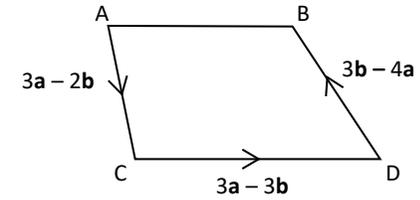
1) Prove \overrightarrow{AB} is parallel to \overrightarrow{CD}



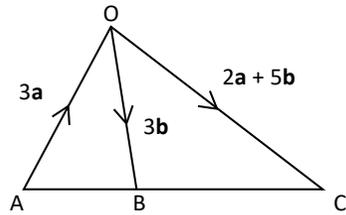
2) Prove \overrightarrow{CD} is parallel to \overrightarrow{AB}



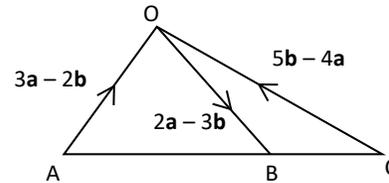
3) Prove \overrightarrow{AB} is parallel to \overrightarrow{CD}



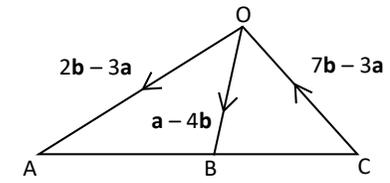
4) Prove \overrightarrow{ABC} is a straight line.



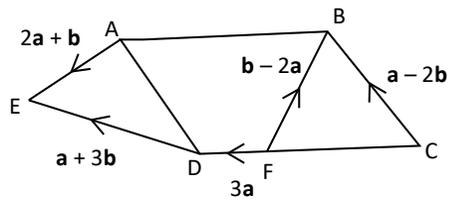
5) Prove \overrightarrow{ABC} is a straight line.



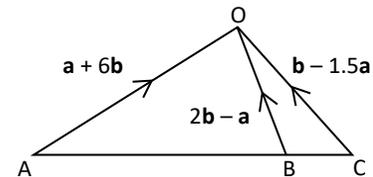
6) Prove \overrightarrow{ABC} is a straight line.



7) Prove ABCD is a parallelogram.



8) Prove \overrightarrow{ABC} is a straight line.



Purposeful Practice

Find the values of λ and μ by equating coefficients of \mathbf{a} and \mathbf{b} .

(a) $4\mathbf{a} + \lambda\mathbf{b} = \mu\mathbf{a} + 6\mathbf{b}$

(b) $\lambda\mathbf{a} - 5\mathbf{b} = -2\mathbf{a} + \mu\mathbf{b}$

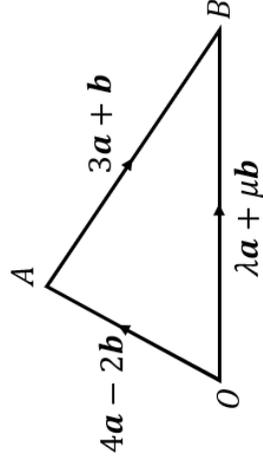
(c) $\lambda\mathbf{a} + \mu\mathbf{b} = 2\mathbf{a} + 6\mathbf{b} + 3\mathbf{a} - 3\mathbf{b}$

(d) $-3\mathbf{a} + \lambda\mathbf{b} = 2\mathbf{a} - \mathbf{b} + \mu\mathbf{a} + 5\mathbf{b}$

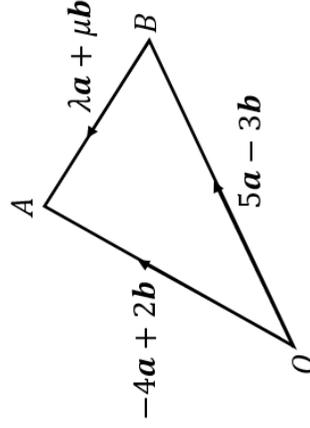
(e) $2(2\mathbf{a} - \mu\mathbf{b}) + 3\mathbf{a} + 10\mathbf{b} = \lambda\mathbf{a} + 6\mathbf{b}$

In the vector diagrams shown, find the values of λ and μ .

(a)

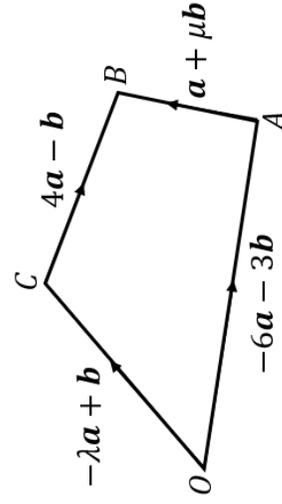


(b)

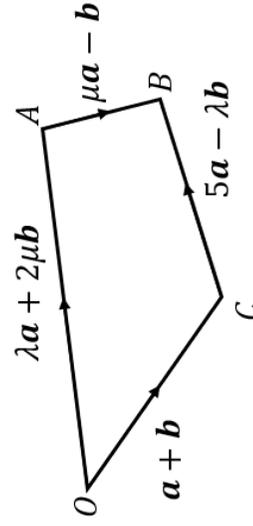


In the vector diagrams shown, find the values of λ and μ .

(a)



(b)

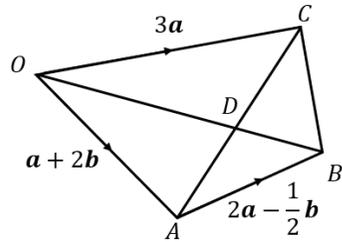


Purposeful Practice

Vector Proof – Equating Coefficients

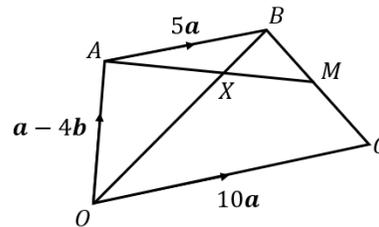
(a)

$OABC$ is a quadrilateral, where $\overrightarrow{OC} = 3\mathbf{a}$, $\overrightarrow{OA} = \mathbf{a} + 2\mathbf{b}$ and $\overrightarrow{AB} = 2\mathbf{a} - \frac{1}{2}\mathbf{b}$. The point D is on OB and AC such that $OD : OB = \lambda : 1$ and $AD : AC = \mu : 1$. By finding two ways to express the vector \overrightarrow{OD} , find the values of λ and μ .



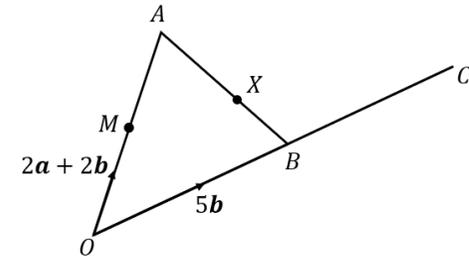
(b)

$OABC$ is a trapezium, where $\overrightarrow{OC} = 10\mathbf{a}$, $\overrightarrow{OA} = \mathbf{a} - 4\mathbf{b}$ and $\overrightarrow{AB} = 5\mathbf{a}$. M is the midpoint of the line BC . The point X is on OB and AM such that $OX : OB = \lambda : 1$ and $AX : AM = \mu : 1$. Find the values of λ and μ and the vector \overrightarrow{OX} in terms of \mathbf{a} and \mathbf{b} .



(c)

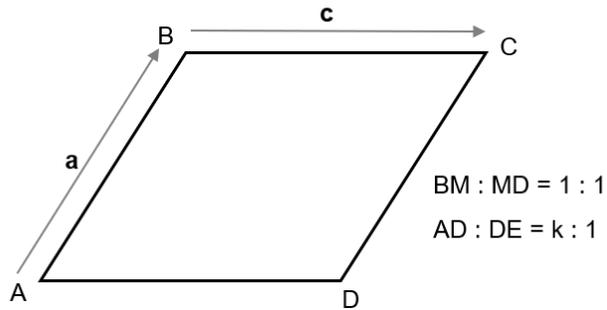
In the triangle OAB , $\overrightarrow{OB} = 5\mathbf{b}$ and $\overrightarrow{OM} = 2\mathbf{a} + 2\mathbf{b}$, where M is the midpoint of OA . OC is the line OB produced and $\overrightarrow{OB} = \overrightarrow{BC}$. The point X is on the line AB such that $AX : AB = \lambda : 1$. Given that MXC is a straight line, find the value of λ and the vector \overrightarrow{MX} in terms of \mathbf{a} and \mathbf{b} .



Purposeful Practice

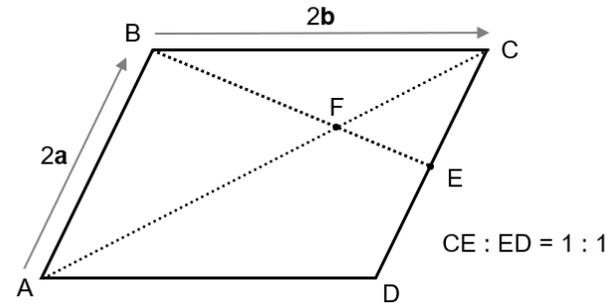
vector proofs

(1) a **parallelogram** (ABCD)



$\vec{ME} = 3\mathbf{c} - \frac{1}{2}\mathbf{a}$ find the value of k

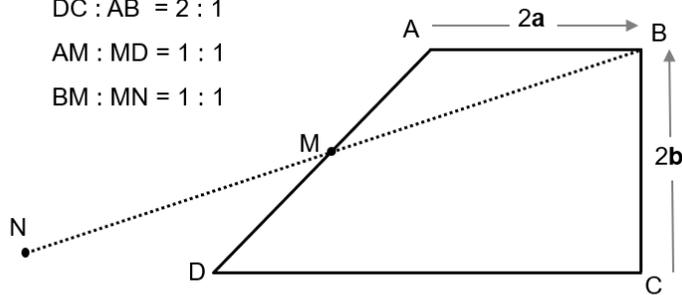
(2) a **parallelogram** (ABCD)



show that $AF : FC = 2 : 1$

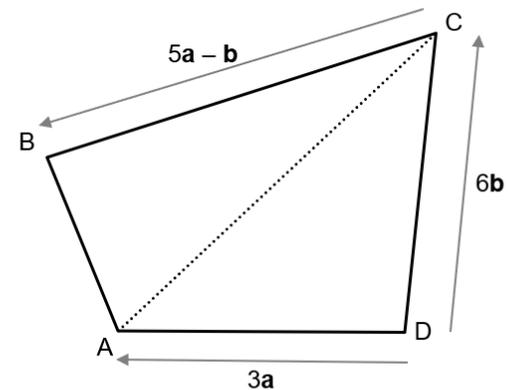
(3) a **trapezium** (ABCD)

$DC : AB = 2 : 1$
 $AM : MD = 1 : 1$
 $BM : MN = 1 : 1$



show that CDN is a straight line

(4)



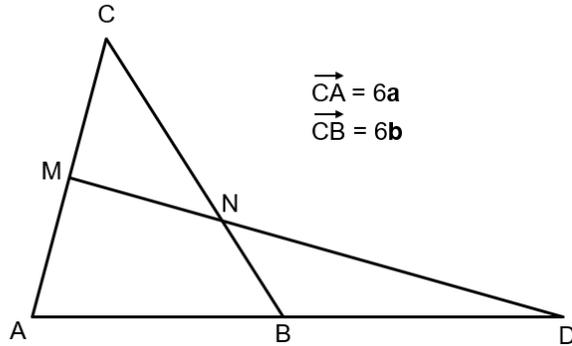
$AX : XC = 1 : 2$

$DX : DB = k : 1$ find the value of k

Purposeful Practice

vector proofs

(1)



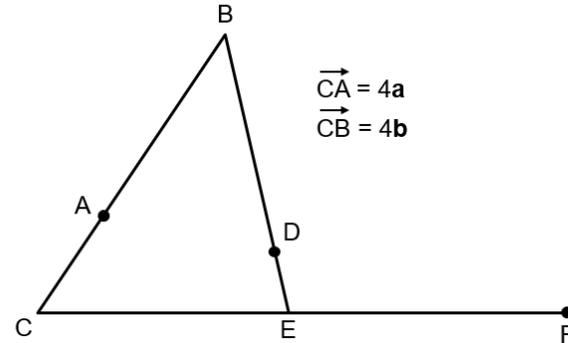
$$\begin{aligned}\vec{CA} &= 6\mathbf{a} \\ \vec{CB} &= 6\mathbf{b}\end{aligned}$$

$$CM : MA = 1 : 1$$

$$AB : BD = 1 : 1$$

$$\vec{CN} = k\mathbf{b} \text{ find the value of } k$$

(2)



$$\begin{aligned}\vec{CA} &= 4\mathbf{a} \\ \vec{CB} &= 4\mathbf{b}\end{aligned}$$

$$CA : AB = 1 : 2$$

$$ED : DB = 1 : 3$$

$$CE : EF = 1 : 2$$

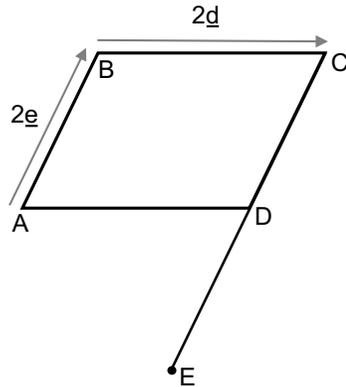
show that ADF is
a straight line

(3) choose some other ratios to explore

Purposeful Practice

vector proofs (i)

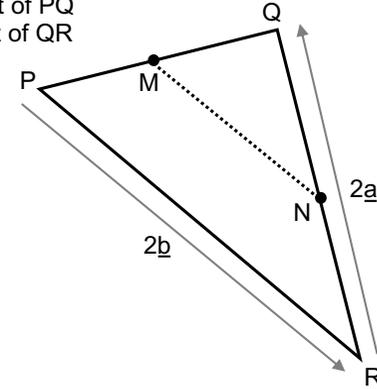
(1) a **parallelogram** (ABCD)



CE is twice CD
establish that
ABDE is a
parallelogram

(2) a **triangle** (PQR)

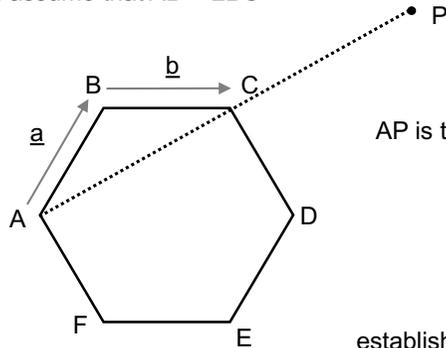
M is the midpoint of PQ
N is the midpoint of QR



prove that MN is
parallel to PR
and half the
length of it

(3) a **regular hexagon**

you can assume that $AD = 2BC$

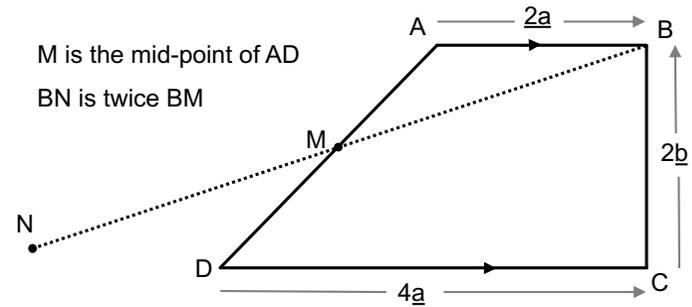


AP is twice AC

establish that EDP
is a straight line

(4) a **trapezium** (ABCD)

M is the mid-point of AD
BN is twice BM

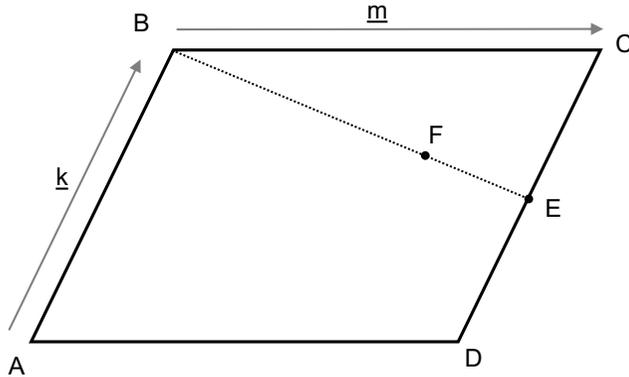


prove that CDN is a straight line

Purposeful Practice

vector proofs (ii)

(4) a **parallelogram** (ABCD)



E is the mid-point of CD

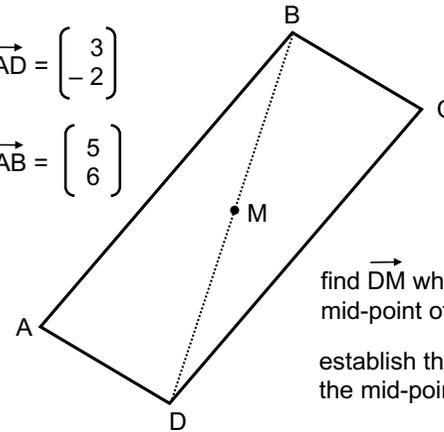
BF : FE = 2 : 1

establish that F is $\frac{2}{3}$ of the way along AC

(5) a **parallelogram** (ABCD)

$$\vec{AD} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\vec{AB} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$



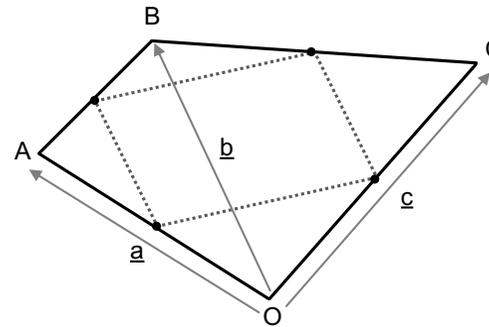
find \vec{DM} where M is the mid-point of DB

establish that M is also the mid-point of AC

(6) any **quadrilateral** (OABC)

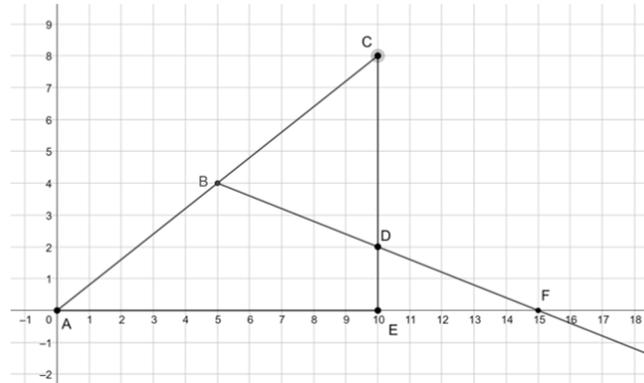
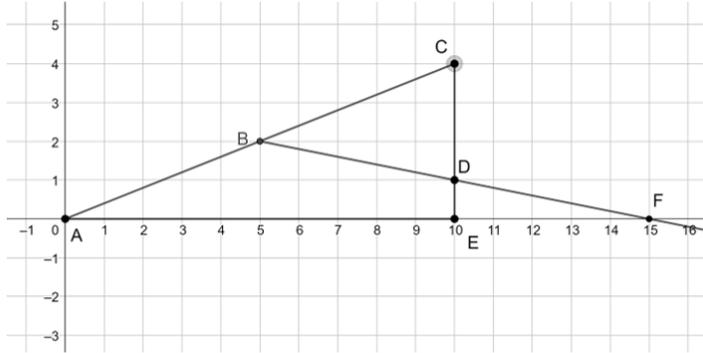
any quadrilateral (OABC) is drawn and the mid-points of the four sides are joined

establish that the resulting shape is a parallelogram



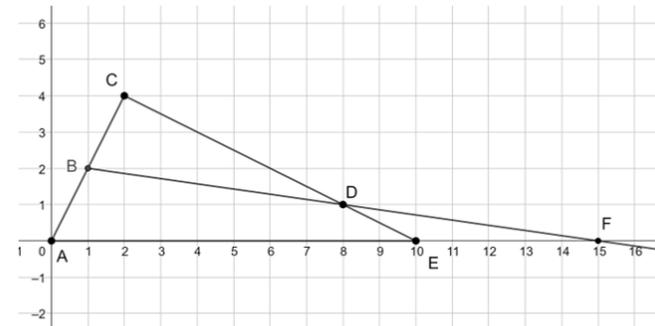
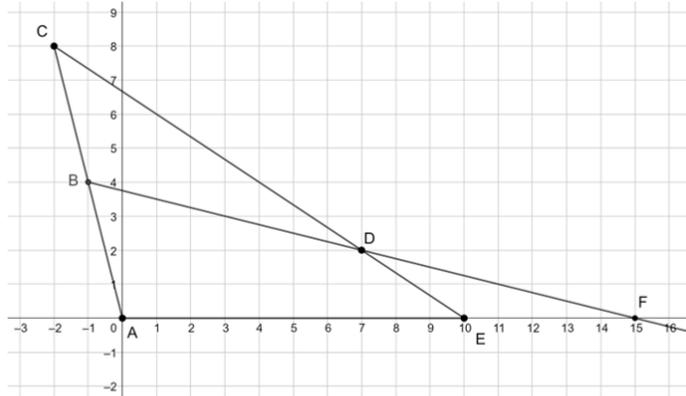
Purposeful Practice

prove the general result



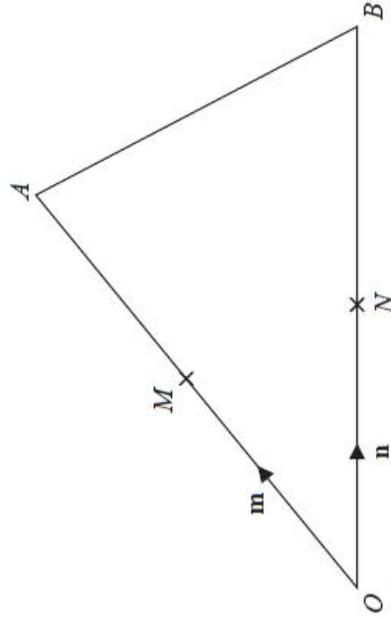
$$\begin{aligned} AB : BC &= 1 : 1 \\ ED : DC &= 1 : 3 \\ AE : EF &= 2 : 1 \end{aligned}$$

prove
 $BD : DF = 1 : 1$



Exam Questions

Q1.



OAB is a triangle.
M is the midpoint of OA.
N is the midpoint of OB.

$$\vec{OM} = \mathbf{m}$$

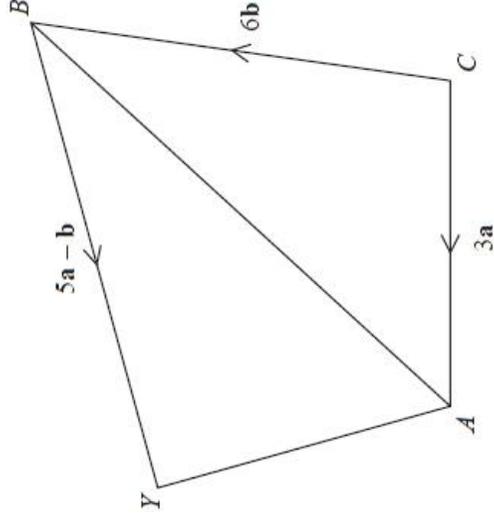
$$\vec{ON} = \mathbf{n}$$

Show that AB is parallel to MN.

(Total for Question is 3 marks)

Exam Questions

Q2.



CAYB is a quadrilateral.

$$\vec{CA} = 3\mathbf{a}$$

$$\vec{CB} = 6\mathbf{b}$$

$$\vec{BY} = 5\mathbf{a} - \mathbf{b}$$

X is the point on AB such that AX : XB = 1 : 2

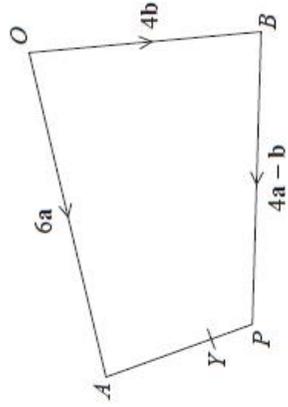
$$\vec{CX} = \frac{2}{5} \vec{CY}$$

Prove that

(Total for question is 5 marks)

Exam Questions

Q3.



$OBPA$ is a quadrilateral.

$$\vec{OA} = 6\mathbf{a}$$

$$\vec{OB} = 4\mathbf{b}$$

$$\vec{BP} = 4\mathbf{a} - \mathbf{b}$$

Y is the point on AP such that $AY : YP = 2 : 1$

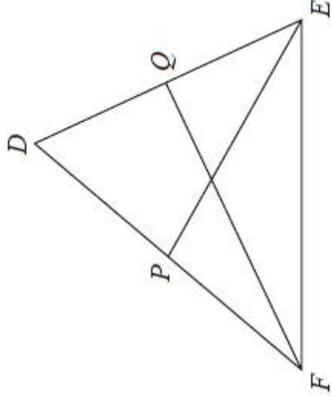
Show that \vec{OY} is parallel to the vector $7\mathbf{a} + 3\mathbf{b}$

(Total for question = 4 marks)

Exam Questions

Q4.

DEF is a triangle.



P is the midpoint of FD .

Q is the midpoint of DE .

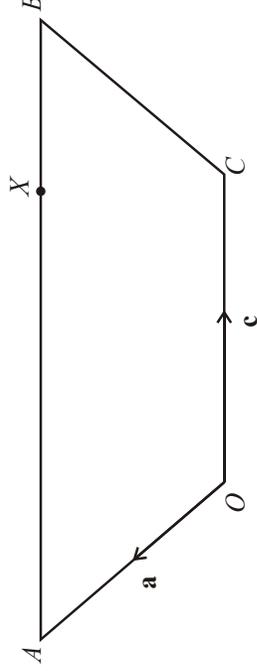
$$\vec{FD} = \mathbf{a} \quad \text{and} \quad \vec{FE} = \mathbf{b}$$

Use a vector method to prove that PQ is parallel to FE .

(Total for question = 4 marks)

Exam Questions

1.



$OACB$ is a trapezium.

OC is parallel to AB .

$$\vec{OA} = \mathbf{a}, \vec{OC} = \mathbf{c}$$

$$AB = 2OC.$$

X is the point on AB such that $AX:XB = 3:1$.

Express \vec{XC} in terms of \mathbf{a} and \mathbf{c} .

2.

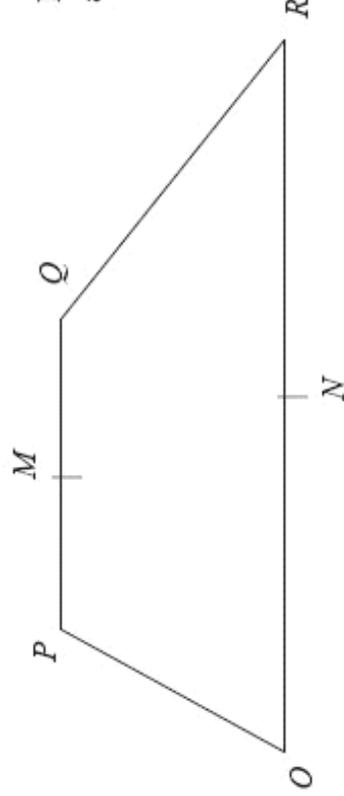


Diagram NOT
accurately drawn

$OPQR$ is a trapezium with PQ parallel to OR .

$$\vec{OP} = 2\mathbf{b} \quad \vec{PQ} = 2\mathbf{a} \quad \vec{OR} = 6\mathbf{a}$$

M is the midpoint of PQ and N is the midpoint of OR .

(a) Find the vector \vec{MN} in terms of \mathbf{a} and \mathbf{b} .

X is the midpoint of MN and Y is the midpoint of QR .

(b) Prove that XY is parallel to OR .

Exam Questions

3.

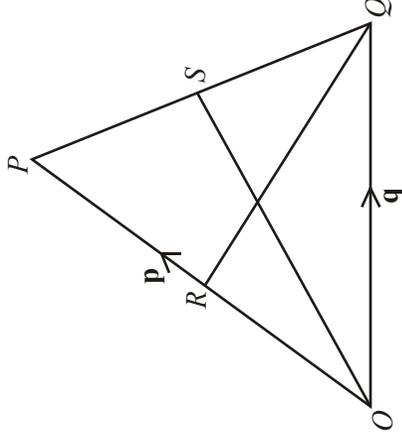


Diagram **NOT**
accurately drawn

OPQ is a triangle.
 R is the midpoint of OP .
 S is the midpoint of PQ .
 $\vec{OP} = \mathbf{p}$ and $\vec{OQ} = \mathbf{q}$

- (i) Find \vec{OS} in terms of \mathbf{p} and \mathbf{q} . (ii) Show that RS is parallel to OQ .

4.

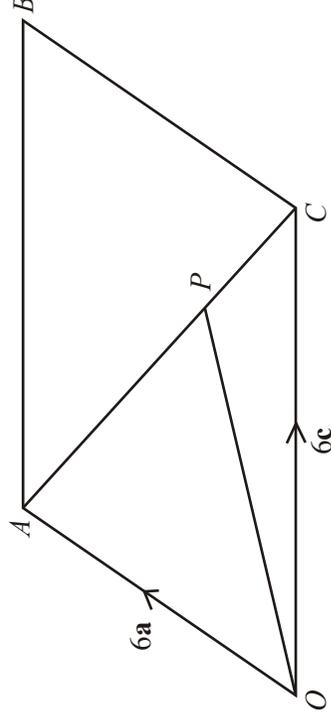


Diagram **NOT**
accurately drawn

$OACB$ is a parallelogram.

P is the point on AC such that $AP = \frac{2}{3}AC$.

$$\vec{OA} = 6\mathbf{a}, \vec{OC} = 6\mathbf{c}.$$

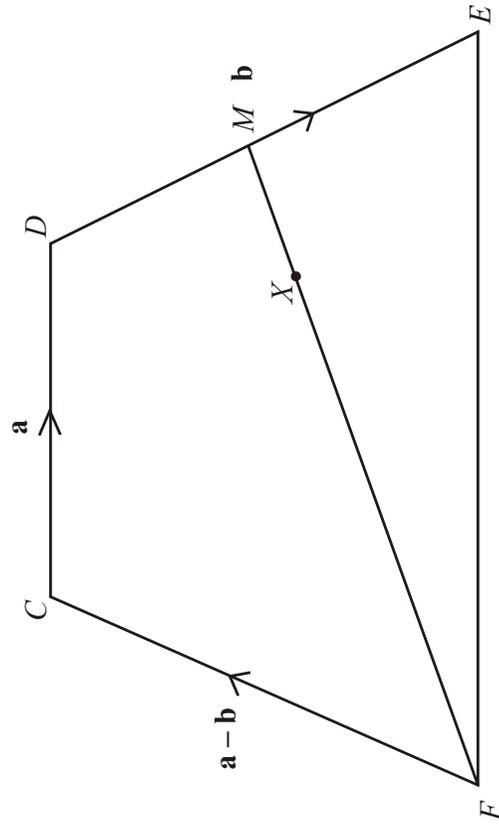
- (a) Find the vector \vec{OP} . Give your answer in terms of \mathbf{a} and \mathbf{c} .

The midpoint of CB is M .

- (b) Prove that OPM is a straight line.

Exam Questions

5.



$CDEF$ is a quadrilateral with $\overrightarrow{CD} = \mathbf{a}$, $\overrightarrow{DE} = \mathbf{b}$ and $\overrightarrow{FC} = \mathbf{a} - \mathbf{b}$.

- (a) Express \overrightarrow{CE} in terms of \mathbf{a} and \mathbf{b} .
- (b) Prove that FE is parallel to CD .
 M is the midpoint of DE .
- (c) Express \overrightarrow{FM} in terms of \mathbf{a} and \mathbf{b} .
- X is the point on FM such that $FX : XM = 4 : 1$.
- (d) Prove that C , X and E lie on the same straight line.

6.

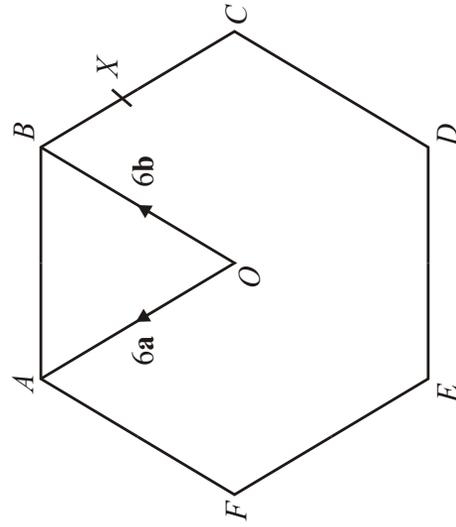


Diagram **NOT**
accurately drawn

The diagram shows a regular hexagon $ABCDEF$ with centre O .

$$\overrightarrow{OA} = 6\mathbf{a} \quad \overrightarrow{OB} = 6\mathbf{b}$$

- (a) Express in terms of \mathbf{a} and/or \mathbf{b}

Exam Questions

- (i) \overrightarrow{AB} ,
- (ii) \overrightarrow{EF} .

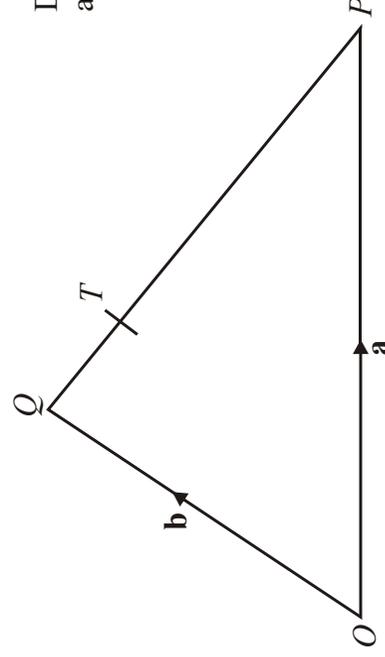
X is the midpoint of BC .

- (b) Express \overrightarrow{EX} in terms of \mathbf{a} and/or \mathbf{b}

Y is the point on AB extended, such that $AB : BY = 3 : 2$

- (c) Prove that E, X and Y lie on the same straight line.

7.



OPQ is a triangle.

T is the point on PQ for which $PT : TQ = 2 : 1$.

$\overrightarrow{OP} = \mathbf{a}$ and $\overrightarrow{OQ} = \mathbf{b}$.

- (a) Write down, in terms of \mathbf{a} and \mathbf{b} , an expression for \overrightarrow{PQ} .
- (b) Express \overrightarrow{OT} in terms of \mathbf{a} and \mathbf{b} .
Give your answer in its simplest form.

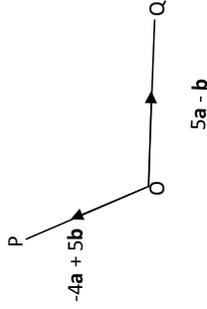
Exam Questions

1. $\vec{OP} = -4\mathbf{a} + 5\mathbf{b}$ and $\vec{OQ} = 5\mathbf{a} - \mathbf{b}$.

(a) Express \vec{PQ} in terms of \mathbf{a} and \mathbf{b} .

R is a point on \vec{PQ} such that $PR : RQ = 1 : 2$.

(b) Express \vec{OR} in terms of \mathbf{a} and \mathbf{b} .



(Total 3 marks)

2. In the diagram $\vec{OP} = 4\mathbf{a}$, $\vec{PA} = \mathbf{a}$,

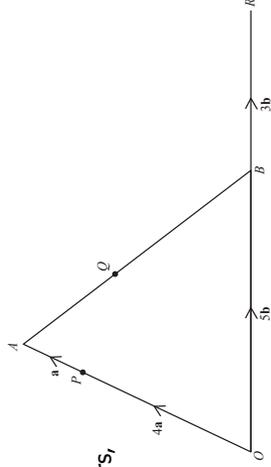
$\vec{OB} = 5\mathbf{b}$, $\vec{BR} = 3\mathbf{b}$ and $\vec{AQ} = \frac{2}{5} \vec{AB}$

(a) Find, in terms of \mathbf{a} and \mathbf{b} , simplifying your answers,

(i) \vec{AB}

(ii) \vec{PQ}

(b) Show clearly that points P , Q and R lie on a straight line.



(Total 6 marks)

3. $ABCDEF$ is a regular hexagon with centre O .

$\vec{OA} = \mathbf{a}$ and $\vec{AB} = \mathbf{b}$

(a) Find expressions, in terms of \mathbf{a} and \mathbf{b} , for

(i) \vec{OB}

(ii) \vec{AC}

(iii) \vec{EC}

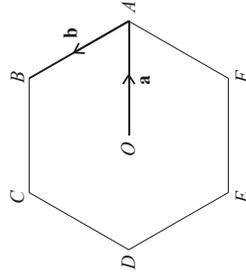


Diagram drawn accurately

(Total 3 marks)

(b) The positions of points P and Q are given by the vectors

$\vec{OP} = \mathbf{a} - \mathbf{b}$ $\vec{OQ} = \mathbf{a} + 2\mathbf{b}$

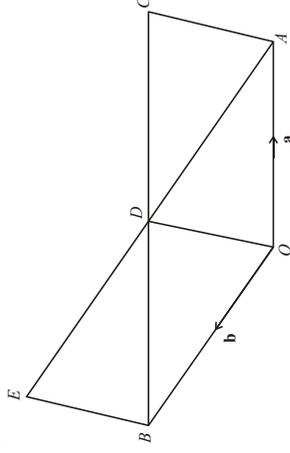
(i) Draw and label the positions of points P and Q on the diagram.

(ii) Hence, or otherwise, deduce an expression for \vec{PQ} .

(Total 6 marks)

Exam Questions

4. In the diagram $OACD$, $OADB$ and $ODEB$ are parallelograms.
 $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$



- (a) Express, in terms of \mathbf{a} and \mathbf{b} , the following vectors.
 Give your answers in their simplest form.

- (i) \overrightarrow{OD}
 (ii) \overrightarrow{OC}
 (iii) \overrightarrow{AB}

- (b) The point F is such that $OCFE$ is a parallelogram.
 Write the vector \overrightarrow{CF} in terms of \mathbf{a} and \mathbf{b} .

- (c) What geometrical relationship is there between the points O , D and F ?
 Justify your answer.

(Total 7 marks)

Not drawn accurately

5. In triangle ABC , M lies on BC such that $BM = \frac{3}{4}BC$.

$$\overrightarrow{AB} = \mathbf{s} \text{ and } \overrightarrow{AC} = \mathbf{t}$$

Find \overrightarrow{AB} in terms of \mathbf{s} and \mathbf{t} .

Give your answer in its simplest form.

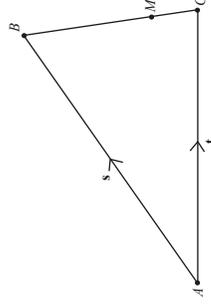
6. In the diagram $\overrightarrow{OP} = 4\mathbf{a}$, $\overrightarrow{PA} = \mathbf{a}$,
 $\overrightarrow{OB} = 5\mathbf{b}$, $\overrightarrow{BR} = 3\mathbf{b}$ and $\overrightarrow{AQ} = \frac{2}{5}\overrightarrow{AB}$

Not drawn accurately

- (a) Find, in terms of \mathbf{a} and \mathbf{b} , simplifying your answers,

- (i) \overrightarrow{AB}
 (ii) \overrightarrow{PQ}

(Total 3 marks)



- (b) Show clearly that points P , Q and R lie on a straight line.

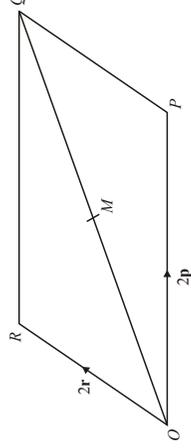
(Total 6 marks)

7. $OPQR$ is a parallelogram.
 M is the mid-point of the diagonal OQ .

$$\overrightarrow{OP} = 2\mathbf{p} \text{ and } \overrightarrow{OR} = 2\mathbf{r}$$

- (a) Express \overrightarrow{OM} in terms of \mathbf{p} and \mathbf{r} .

- (b) Use vectors to prove that M is also the mid-point of PR .



(Total 4 marks)

Exam Questions

8. $PQRSTU$ is a regular hexagon and O is the centre of the hexagon.

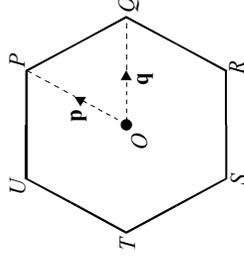
$$\vec{OP} = \mathbf{p} \text{ and } \vec{OQ} = \mathbf{q}$$

Express each of the following vectors in terms of \mathbf{p} and \mathbf{q}

(a) \vec{PQ}

(b) \vec{SP}

(c) \vec{SQ}



9. The diagram shows two sets of parallel lines.
Vector $\vec{PQ} = \mathbf{a}$ and vector $\vec{PS} = \mathbf{b}$

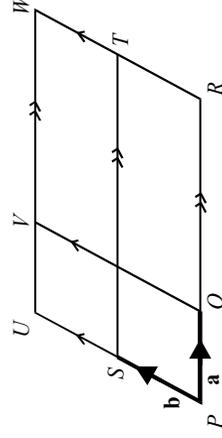
$$\vec{PR} = 3\vec{PQ} \text{ and } \vec{PU} = 2\vec{PS}$$

(a) Write the vector \vec{PV} in terms of \mathbf{a} and \mathbf{b}

(b) Write the vector \vec{RU} in terms of \mathbf{a} and \mathbf{b}

(c) Find **two** vectors that can be written as $3\mathbf{a} - \mathbf{b}$

(Total 4 marks)



10. $OACB$ is a parallelogram and M is the mid-point of BC .

$$\vec{OA} = \mathbf{a} \text{ and } \vec{OB} = \mathbf{b}$$

(a) Express the following vectors in terms of \mathbf{a} and \mathbf{b}

(i) \vec{BA}

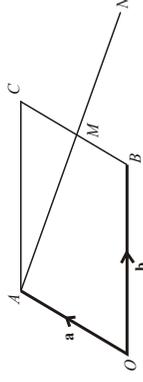
(iii) \vec{AM}

(b) AM is extended to N , where $\vec{AN} = 2\vec{AM}$.

$$\text{Show that } \vec{BN} = \mathbf{b}$$

(c) What does this tell you about the position of N ?

(Total 4 marks)



11. In triangle ABC , M is the mid-point of BC .

$$\vec{AB} = \mathbf{s} \text{ and } \vec{AC} = \mathbf{t}$$

(a) Find \vec{AM} in terms of \mathbf{s} and \mathbf{t} .

Give your answer in its simplest form.

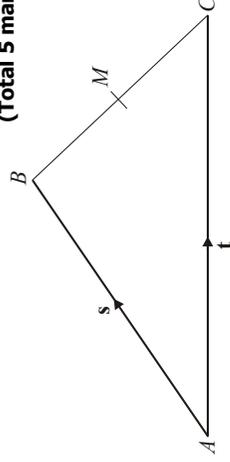
(b) $\vec{AD} = \mathbf{s} + \mathbf{t}$

The length of AB is **not** equal to the length of AC .

(i) Write down the name of the shape $ABDC$.

(ii) Write down one fact about the points A , M and D .
Explain your answer.

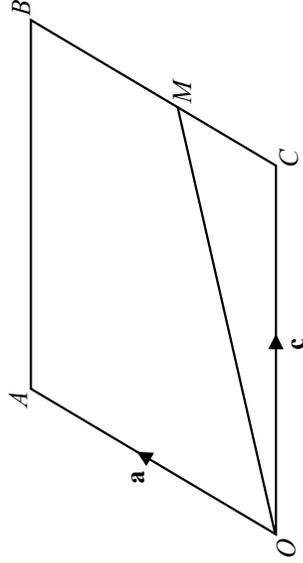
(Total 5 marks)



(Total 6 marks)

Exam Questions

1 $OABC$ is a parallelogram.



$$\vec{OA} = \mathbf{a} \quad \vec{OC} = \mathbf{c}$$

M is the point on BC such that $BM : MC = 3 : 2$

The line AB is extended to the point N .

OMN is a straight line.

Find the ratio $AB : BN$.

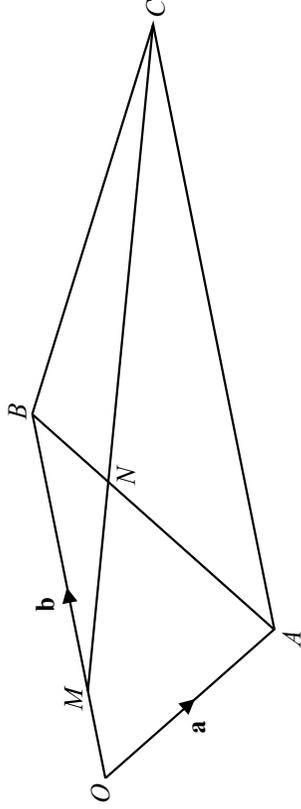
Give your answer in the form $k : 1$

$AB : BN = \dots\dots\dots$

(Total for Question 1 is 4 marks)

Exam Questions

2 $OABC$ is a quadrilateral.



$$\vec{OA} = \mathbf{a} \quad \vec{OB} = \mathbf{b}$$

MNC and ANB are straight lines.

M is the point on the line OB such that $OM : MB = 1 : 5$

N is the point such that $AN : NB = 3 : 2$

\vec{OB} is parallel to AC .

Work out the ratio $OB : AC$.

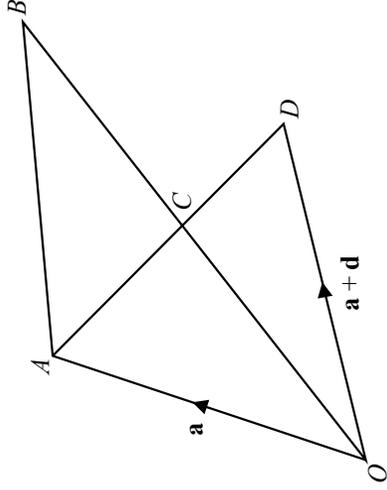
Give your answer in the form $k : 1$

$OB : AC = \dots\dots\dots$

(Total for Question 2 is 5 marks)

Exam Questions

3 OAD is a triangle.



$$\vec{OA} = \mathbf{a} \quad \vec{OD} = \mathbf{a} + \mathbf{d}$$

C is the point on the line AD such that $AC : AD = 2 : 3$
 OCB is a straight line such that $OC : CB = 3 : 5$

$$\vec{AB} = \lambda \mathbf{a} + \mu \mathbf{d}$$

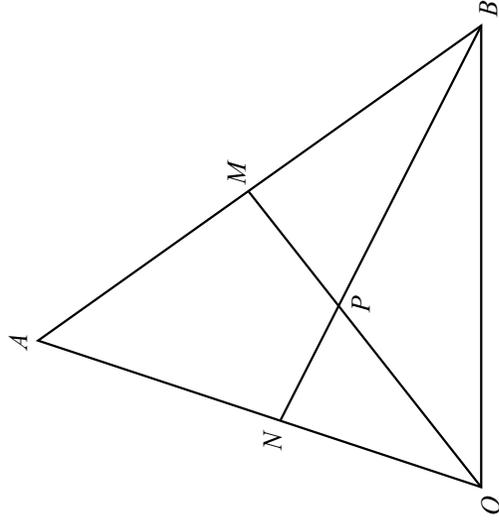
Work out the ratio $\lambda : \mu$
 Give your answer as a ratio in its simplest form.

$\lambda : \mu = \dots\dots\dots$

(Total for Question 3 is 4 marks)

Exam Questions

- 4 OAB is a triangle.



$$\vec{OA} = \mathbf{a} \quad \vec{OB} = \mathbf{b}$$

M is the midpoint of AB .

OPM and OPN are straight lines.

P is the point such that $OP : PM = 4 : 3$

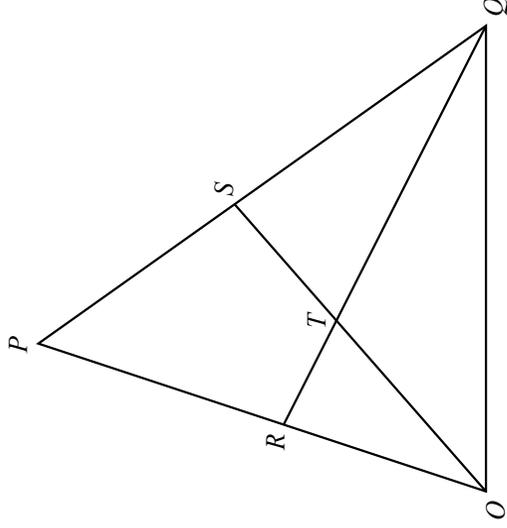
Find the ratio $ON : NA$

$$ON : NA = \dots\dots\dots$$

(Total for Question 4 is 5 marks)

Exam Questions

5 OPQ is a triangle.



$$\vec{OP} = \mathbf{p} \quad \vec{OQ} = \mathbf{q}$$

OTS and RTQ are straight lines.

R is the point on OP such that $OR : RP = 3 : 4$

T is the point such that $TQ : RQ = 3 : 4$

Find the ratio $PS : SQ$

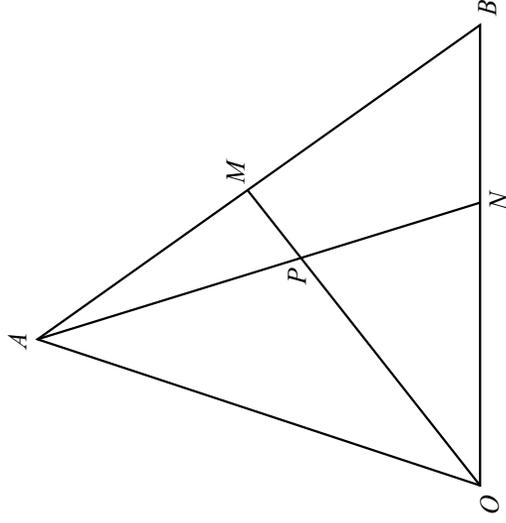
Give your answer in the form $a : b$ where a and b are integers.

$PS : SQ = \dots\dots\dots$

(Total for Question 5 is 5 marks)

Exam Questions

6 OAB is a triangle.



OPM and APN are straight lines.
 M is the midpoint of AB .

$$\vec{OA} = \mathbf{a} \quad \vec{OB} = \mathbf{b}$$

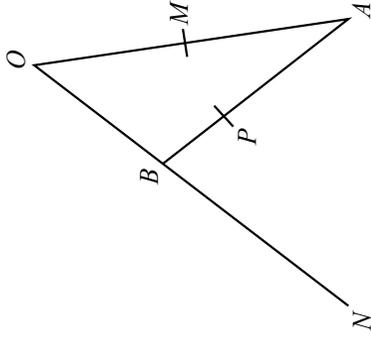
$$OP : PM = 5 : 2$$

Work out the ratio $ON : NB$

.....
(Total for Question 6 is 5 marks)

Exam Questions

7



OBN , OMA and APB are straight lines.

$BN = 2OB$.

M is the midpoint of OA .

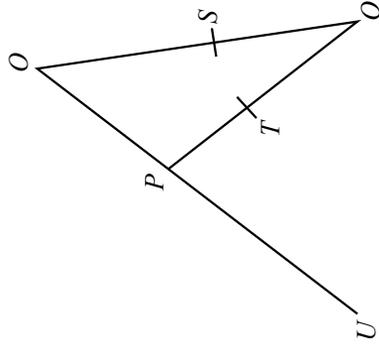
$$\vec{OA} = \mathbf{a} \quad \vec{OB} = \mathbf{b}$$

Given that MPN is a straight line, work out the ratio $AP : PB$

.....
(Total for Question 7 is 5 marks)

Exam Questions

8



OPU , OSQ and PTQ are straight lines.

$$OP : PU = 3 : 4$$

$$OS : OQ = 5 : 8$$

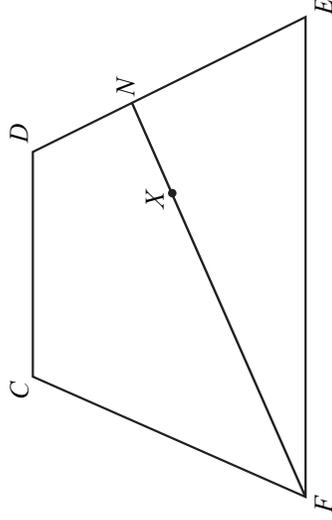
$$\vec{OP} = \mathbf{p} \quad \vec{OQ} = \mathbf{q}$$

Given that STU is a straight line, work out the ratio $PT : TQ$

.....
(Total for Question 8 is 5 marks)

Exam Questions

9 $CDEF$ is a quadrilateral.



$$\vec{CD} = \mathbf{a}, \vec{DE} = \mathbf{b} \text{ and } \vec{FC} = \mathbf{a} - \mathbf{b}.$$

N is the point on DE such that $DN : NE = 2 : 3$
 X is the point on FN such that $FX : XN = k : 1$
 CXE is a straight line.

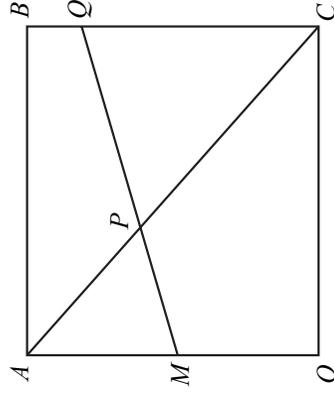
Work out the value of k .

$k =$

(Total for Question 9 is 5 marks)

Exam Questions

- 10 $OABC$ is a square.



$$\vec{OA} = \mathbf{a}, \vec{OC} = \mathbf{c}.$$

M is the midpoint of OA .

Q divides BC in the ratio $BQ : QC = 1 : 3$

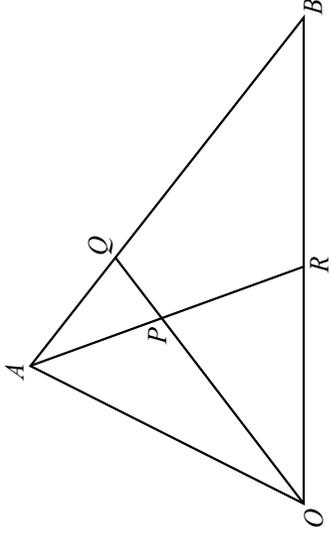
APC and MPQ are straight lines.

Show that P divides AC in the ratio $2 : 3$

(Total for Question 10 is 5 marks)

Exam Questions

11 OAB is a triangle.



$$\vec{OA} = \mathbf{a} \quad \vec{OB} = \mathbf{b}$$

OPQ and APR are straight lines.

The point P divides AB in the ratio $AP : PR = 3 : 4$

The point Q divides AQB in the ratio $AQ : QB = 2 : 5$

Find the ratio $OR : RB$

Give your answer in its simplest form.

.....
(Total for Question 11 is 6 marks)