KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 11

2025 Mathematics (L2FM) 2026

Unit 26 Tasks

DO NOT WRITE INSIDE

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1 Differentiation (L2FM Only)

Fluency Practice

Find the gradient function $\frac{dy}{dx}$ when:

(a) $y = x^4$ (b) $y = x^9$

(c) $y = x^7$ (d) $y = x^6$

(e) $y = x$ (f) $y = x^{10}$

Find the gradient function $\frac{dy}{dx}$ when:

(a) $y = 7x^2$

(b) $y = 3x^5$

(c) $y = 10x^6$

(d) $y = 2x^9$

(e) $y = \frac{1}{2}x^8$

(f) $y = \frac{1}{5}x^4$

(g) $y = 0.3x^5$

(h) $y = -6x^3$

Find the gradient function $\frac{dy}{dx}$ when:

(a) $y = x^2 + x^5$

(b) $y = 3x^2 + 7x^5$

(c) $y = 5x^4 - x^3$

(d) $y = 2x^3 - x^2 + 5x$

(e) $y = 3x + 6x^4$

(f) $y = 0.5x^7 + 3$

(g) $y = \frac{1}{4}x^5 - x^3 + 7x$

(h) $y = x^3 + 2x^2 - 7x + 10$

(a) Expand and simplify $(x + 3)(x^2 - 5)$

(b) Hence find the gradient function $\frac{dy}{dx}$ when $y = (x + 3)(x^2 - 5)$

Fluency Practice

Find $\frac{dy}{dx}$ when:

(a) $y = x^3(x + 2)$

(b) $y = 2x(x^5 - 4x^3)$

(c) $y = (x + 7)(x - 3)$

(d) $y = (3x - 5)(2x + 1)$

(e) $y = (x^2 + 3)(x - 5)$

(f) $y = x(x + 4)(x - 4)$

Find $\frac{dy}{dx}$ when:

(a) $y = \frac{8x^5+6x^2}{2}$

(b) $y = \frac{x^4 - 2x^3}{x}$

(c) $y = \frac{10x^4 - 5x^3}{2x}$

(d) $y = \frac{9x^7 + 2x^3}{3x^2}$

(e) $y = \frac{4x^2(x-7)}{2x}$

Find $\frac{dy}{dx}$ when:

(a) $y = \frac{7}{x}$ (b) $y = -\frac{3}{x^2}$

(c) $y = \frac{5}{2x}$ (d) $y = \frac{4}{5x^3}$

(e) $y = 2x^5 + x^3 - \frac{3}{x}$

(f) $y = 7x^2 + 4x + \frac{5}{2x}$

(g) $y = 6x^3 + \frac{1}{x} - \frac{5}{x^2}$

(h) $y = (x + 3)\left(x + \frac{1}{x}\right)$

(i) $y = \frac{10x^4+4x^2+2}{2x}$

Fluency Practice

1 For each of the following, find the gradient function $\frac{dy}{dx}$, and hence find the gradient of the tangent to the curve when $x = 2$

a $y = 2x^5$ → ?

b $y = 7x^3$ → ?

c $y = 3x + 4$ → ?

d $y = 2x + x^2$ → ?

e $y = 2x^3 + x^{-1}$ → ?

f $y = \frac{1}{2}x^{-2}$ → ?

g $y = 2\pi x^2 - 1$ → ?

2 The tangent to the curve $y = x^3 + 2x^2 - x$ has gradient 6. Determine the possible values of x.

?

3 For the curve $y = x^2 + x - 2$, determine:

(a) The gradient of the tangent to the curve at the point $(1,0)$

?

(b) The point on the curve where the gradient is 5.

?

4 Find the points on the curve $y = \frac{1}{3}x^3 + x^2 - 4x$ where the gradient is 11.

?

Purposeful Practice

① Differentiate the following functions.

(i) $y = x^4$ (ii) $y = 2x^3$ (iii) $y = 5x^2$
(iv) $y = 7x^9$ (v) $y = -3x^6$ (vi) $y = 5$
(vii) $y = 10x$ (viii) $y = \frac{1}{4}x^3$ (ix) $y = 2\pi x$
(x) $y = \pi x^2$

② Differentiate the following functions.

(i) $y = 2x^5 + 4x^2$ (ii) $y = 3x^4 + 8x$ (iii) $y = x^3 + 4$
(iv) $y = x - 5x^3$ (v) $y = 4x^3 + 2x$ (vi) $y = 2x + 6$
(vii) $y = 3x^5 + 2$

③ Differentiate the following functions.

(i) $y = 3x^5 + 4x^4 - 3x^2 + 2$
(ii) $y = x^5 + 12x^3 + 3x$
(iii) $y = x^3 + 42x^2 - 5x + 24$

④ Write down the rate of change of the following functions with respect to y.

(i) $y = x^{-4}$ (ii) $y = 3x^{-2}$ (iii) $y = 3x^2 + 4x^{-1}$
(iv) $y = 2x^{-3} - 4$ (v) $y = x^2 + x^{-2}$ (vi) $y = 3x^{-2} + 2x^{-3}$

⑤ Differentiate the following functions.

(i) $y = 3x^2 + \frac{2}{x^3}$ (ii) $y = x^2 + \frac{1}{x^2}$ (iii) $y = 3x^3 + \frac{3}{x^3}$
(iv) $y = \frac{2}{x} - \frac{3}{x^2}$ (v) $y = \frac{1}{2x} - \frac{1}{3x^2}$ (vi) $y = \frac{2}{3x} - \frac{3}{4x^2}$

⑥ A rectangle has length $6x$ and width $3x$.
The area of the rectangle is y.

(i) Write down y in terms of x.
(ii) Work out $\frac{dy}{dx}$.

Figure 8.6

⑦ When a stone is thrown into a lake circular ripples appear centred on the point at which the stone entered the water and spreading outwards. After a time, t seconds, the radius of the circle is r cm where $r = 10t^2$.

(i) Work out the rate at which the radius is increasing (include the units).

With time, the definition of the ripples becomes negligible so that after 8 seconds they cannot be seen by the human eye.

(ii) What is the area of the largest ripple that you can see? Give your answer to the nearest 10 square metres.

⑧ An expanding sphere has radius $2x$.

(i) Show that the volume, y, of the sphere is given by the formula $y = \frac{32}{3}\pi x^3$.
(ii) Work out the rate of change of y with respect to x when $x = 2$

Fluency Practice

① Work out the gradient function for each of the following functions.

(i) $y = x(x^2 + 2)$ (ii) $y = 2x^2(3x - 4)$

(iii) $y = (x + 3)(x + 2)$ (iv) $y = (x + 5)(x + 2)$

(v) $y = x^3(4 + x - x^2)$ (vi) $y = (x + 2)(x - 5)$

② Work out an expression for the rate of change of y with respect to x for each of the following.

(i) $y = \dfrac{x^5 + x^3}{4}$ (ii) $y = \dfrac{x^7 + x^3}{x^2}$

(iii) $y = \dfrac{4x^6 - 2x^2}{x^2}$ (iv) $y = (3x + 1)(x - 2)$

(v) $y = x^{\frac{1}{2}}\left(x^{\frac{3}{2}} + x^{\frac{1}{2}}\right)$ (vi) $y = x^{\frac{1}{2}}\left(x^{\frac{7}{2}} + x^{-\frac{1}{2}}\right)$

③ (i) Simplify $\dfrac{3x^3 - 2x^2}{x}$.

(ii) Use your answer to (i) to differentiate $y = \dfrac{3x^3 - 2x^2}{x}$.

④ Work out the gradient of the curve $y = x^3(x - 2)$ at the point $(3, 27)$.

⑤ Work out the rate of change of y with respect to x for $\dfrac{6x^4 + 2x^5}{2x^3}$ when $x = -1$

⑥ Work out the rate of change of y with respect to x for $y = x^{\frac{1}{3}}\left(x^{\frac{5}{3}} - x^{\frac{2}{3}}\right)$ when $x = -3$

⑦ Work out the gradient of the curve $y = \dfrac{3x^4 + x^2 - 5x}{x}$ at the point $(1, -1)$.

⑧ Work out the gradient of the curve $y = 3\sqrt{x} - \dfrac{3}{\sqrt{x}}$ at the point $(4, 4.5)$.

Note

The rule

$y = x^n \Rightarrow \dfrac{dy}{dx} = nx^{n-1}$

is valid for all values of n but will only be examined when n is an integer.

Fluency Practice

(a) Find the equation of the tangent to the curve $y = x^2 + 2x - 3$ at the point $(1, 0)$.

(b) Find the equation of the tangent to the curve $y = x^2 + 4x - 5$ at the point $(-1, -7)$.

(c) Find the equation of the tangent to the curve $y = x^3 + x$ at the point $(2, 10)$.

(a) Find the equation of the normal to the curve $y = x^2 - 4$ at the point $(1, -3)$.

(b) Find the equation of the normal to the curve $y = x^2 - 5x - 6$ at the point $(3, -12)$.

(c) Find the equation of the normal to the curve $y = 2x^3 - 3x + 1$ at the point $(1, 0)$.

(a) Find the equation of the tangent to the curve $y = x^2 + \frac{1}{x}$ at the point where $x = 1$.

(b) Find the equation of the normal to the curve $y = x(x + 2)(x - 1)$ at the point where $x = -2$.

(a) Find the equation of the tangent to the curve $y = 3x - x^2$ at the point $x = 2$.

(b) The tangent crosses the x-axis and y-axis at A and B respectively. Find the area of the triangle AOB.

Fluency Practice

1 [IGCSEFM June 2012 Paper 1 Q8] A curve has equation $y = x^3 + 5x^2 + 1$

(a) When $x = -1$, show that the value of $\frac{dy}{dx}$ is -7.

?

(b) Work out the equation of the tangent to the curve $y = x^3 + 5x^2 + 1$ at the point where $x = -1$.

?

2 [IGCSEFM June 2013 Paper Q8] A curve has equation $y = x^4 - 5x^2 + 9$

(a) Work out $\frac{dy}{dx}$. $\frac{dy}{dx} =$?

(b) Work out the equation of the tangent to the curve at the point where $x = 2$

Give your answer in the form $y = mx + c$

?

3 [IGCSEFM Set Paper 1 Q11] Show that the tangents to the curve $y = x^3 + 3x^2 + 3x + 1$ at $x = 1$ and $x = -3$ are parallel.

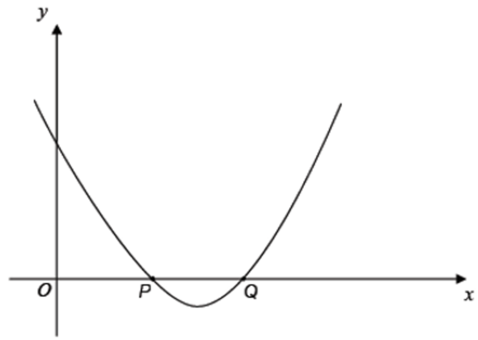
?

4 [IGCSEFM Set 1 Paper 2 Q17] Work out the equation of the normal to the curve $y = 2x^3 - x^2 + 1$ at the point (1, 2). Give your answer in the form $y = mx + c$.

?

Fluency Practice

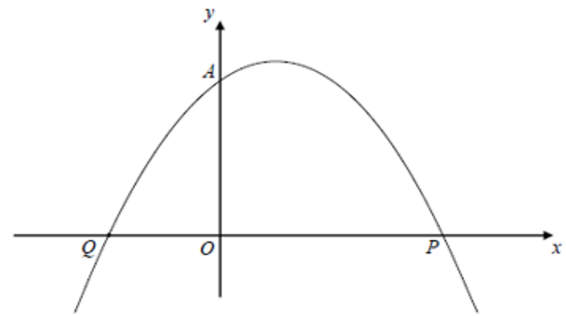
5 [IGCSEFM Set 2 Paper 1 Q15] The graph shows a sketch of $y = (x-2)(x-3)$. The curve intersects the x-axis at P and Q.



Show that the tangents at P and Q are perpendicular.

?

6 [IGCSEFM Set 4 Paper 2 Q20] A sketch of the curve $y = (x+1)(2-x)$ is shown. $A(0,2), P(2,0)$ and Q are points on the curve.

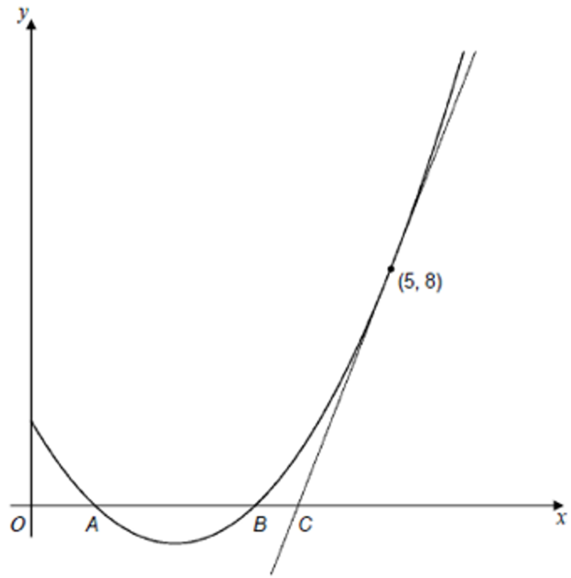


(a) Write down the coordinates of point Q.
(b) Show that the normal to the curve at A intersects the curve again at P.

?

Fluency Practice

7



[IGCSEFM Specimen Paper 2 Q22] The diagram shows the graph of $y = x^2 - 4x + 3$
The curve cuts the x-axis at the points A and B.
The tangent to the curve at the point (5,8) cuts the x-axis at the point C.
Show that $AB = 3BC$.

?

Purposeful Practice

① The sketch shows the graph of $y = 5x - x^2$.

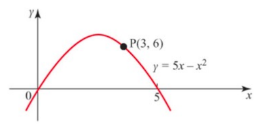


Figure 8.11

The marked point, P, has coordinates (3, 6). Work out

(i) the gradient function $\frac{dy}{dx}$

(ii) the gradient of the curve at P

(iii) the equation of the tangent at P

(iv) the equation of the normal at P.

② The sketch shows the graph of $y = 3x^2 - x^3$.

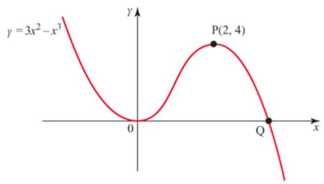


Figure 8.12

(i) The marked point, P, has coordinates (2, 4). Work out

(a) the equation of the tangent at P

(b) the equation of the normal at P.

(ii) The graph touches the x-axis at the origin O and crosses it at the point Q. Work out the equation of the tangent at Q.

(iii) Without further calculation, state the equation of the tangent to the curve at O.

③ The sketch shows the graph of $y = x^5 - x^3$.

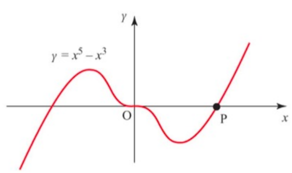


Figure 8.13

(i) Work out the coordinates of the point P where the curve crosses the positive x-axis.

(ii) Work out the equation of the tangent at P.

(iii) Work out the equation of the normal at P.

The tangent at P meets the y-axis at Q and the normal meets the y-axis at R.

(iv) Work out the coordinates of Q and R and hence calculate the area of triangle PQR.

④ (i) Given that $y = x^3 - 3x^2 + 4x + 1$, work out the gradient function $\frac{dy}{dx}$.

(ii) The point P is on the curve $y = x^3 - 3x^2 + 4x + 1$ and its x-coordinate is 2.

(a) Work out the equation of the tangent at P.

(b) Work out the equation of the normal at P.

(iii) Work out the values of x for which the curve has a gradient of 13

⑤ The sketch shows the graph of $y = x^3 - 9x^2 + 23x - 15$

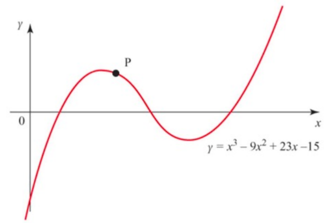


Figure 8.14

The point P marked on the curve has its x-coordinate equal to 2

(i) Work out the equation of the tangent at P.

Q is a point on the curve where the tangent is parallel to the tangent at P.

(ii) Work out the equation of the tangent at Q.

⑥ The point (2, −8) is on the curve $y = x^3 - px + q$.

(i) Identify a relationship between p and q.

The tangent to this curve at the point (2, −8) is parallel to the x-axis.

(ii) Work out the value of p.

(iii) Work out the coordinates of the other point where the tangent is parallel to the x-axis.

(iv) Work out the equation of the normal to the curve at the point where it crosses the y-axis.

⑦ The sketch shows the graph of $y = x^2 - x - 1$

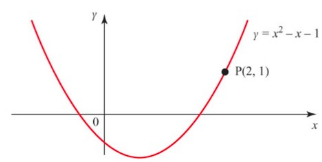


Figure 8.15

(i) Work out the equation of the tangent at the point P.

The normal at a point Q on the curve is parallel to the tangent at P.

(ii) Work out the coordinates of the point Q.

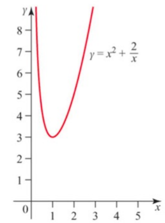


Figure 8.16

⑧ A curve has the equation $y = (x - 3)(7 - x)$.

(i) Work out the equation of the tangent at the point (6, 3).

(ii) Work out the equation of the normal at the point (6, 3).

(iii) Which one of these lines passes through the origin?

⑨ A curve has the equation $y = 1.5x^3 - 3.5x^2 + 2x$.

(i) Show that the curve passes through the points (0, 0) and (1, 0).

(ii) Work out the equations of the tangents and normals at each of these points.

(iii) What shape is formed by the four lines in part (ii)?

⑩ Figure 8.16 shows the curve with the equation $y = x^2 + \frac{2}{x}$ for $x > 0$

(i) Work out the gradient function $\frac{dy}{dx}$ and calculate the coordinates of the minimum point.

(ii) State the equations of the tangent and the normal at that minimum point.

(iii) Work out the equations of the tangent and normal at the point where $x = 2$

Fluency Practice

1 $y = \frac{1}{3}x^3 + 3x^2 + 10x + 1$

a) Find $\frac{dy}{dx}$?

b) By completing the square, hence show that this function is increasing for all x.

?

2 $f(x) = x^2 + 3x + 5$

a) Find $f'(x)$?

b) Hence or otherwise, determine the values of x for which $f(x)$ is increasing.

?

3 $y = x^2 - x + 2$

Determine the values of x for which y is decreasing.

?

4 $y = x^3 - 7x^2 - 5x$

Determine the values of x for which y is increasing.

?

5 Show that $f(x) = x^3 - 9x^2 + 39x - 4$ is an increasing function for all values of x.

?

6 [AQA IGCSEFM June 2012 Paper 2 Q20] For what values of x is $y = 150x - 2x^3$ an increasing function?

?

7 [Set 4 Paper 1 Q10] $y = 10 - 8x - x^3$ for all values of x. Show that y is a decreasing function for all values x.

?

Purposeful Practice

① Work out the values of x for which the following functions are increasing.

(i) $y = x^2 + 4$
(ii) $y = 2x - 3$
(iii) $y = x^2 + 2x - 5$
(iv) $y = x^2 - 3x$
(v) $y = 3x^2 + 4x + 7$
(vi) $y = (x + 6)(x - 2)$
(vii) $y = x^3 - 2x^2$
(viii) $y = x^3 + 6x^2 - 15x$
(ix) $y = x^3 - 3x^2 - 9x + 1$

② Work out the values of x for which the following functions are decreasing.

(i) $y = 4x^2$
(ii) $y = x^2 - 6x + 2$
(iii) $y = x(x + 2)$
(iv) $y = 3 + 4x - x^2$
(v) $y = 12 - x$
(vi) $y = (2x + 1)^2$
(vii) $y = \frac{1}{3}x^3 + x^2$
(viii) $y = 2x^3 - 3x^2 - 72x$
(ix) $y = 27x - x^3$

③ Prove that $y = \frac{1}{3}x^3 + 2x^2 + 7x + 1$ is an increasing function for all values of x.

④ Prove that $y = x^3 - 6x^2 + 27x - 4$ is an increasing function for all values of x.

⑤ Work out the values of x for which $y = x^2 + \frac{2}{x}$ is an increasing function.

⑥ Prove that $y = 12 - 2x - x^3$ is a decreasing function for all values of x.

⑦ Prove that $y = \frac{1}{x}$ is a decreasing function for all $x \neq 0$

⑧ Work out the values of x for which the following functions are

(a) increasing
(b) decreasing.

(i) $y = x + \frac{1}{x}$
(ii) $y = x - \frac{1}{x}$
(iii) $y = x^2 + \frac{1}{x^2}$
(iv) $y = x^2 - \frac{1}{x^2}$

⑨ Air is being pumped into a spherical balloon at the rate of $1000\,\text{cm}^3\,\text{s}^{-1}$. Initially the balloon contains no air. (The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$).

(i) Calculate the volume V of the balloon after 10 seconds.
(ii) Calculate the volume of the balloon after t seconds.
(iii) State the value of $\frac{dV}{dt}$.
(iv) Calculate the radius of the balloon after t seconds.

Purposeful Practice

① Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions.

(i) $y = 3x^3 + 3x$ (ii) $y = x^5 - 25$

(iii) $y = 3x - 5x^4$

② Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions.

(i) $y = x^4 - 2x^2 + 5x - 4$ (ii) $y = 2x^3 + 3x - 4$

(iii) $y = x^3 - 2x^2 + 1$

③ Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions. Remember that when an expression involves brackets you need to multiply out before differentiating.

(i) $y = (2x - 1)(x + 2)$ (ii) $y = (2x - 1)^2$

(iii) $y = (1 - 3x)(2x - 3)$

④ Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions.

(i) $y = 3(x - 2)(x^2 - 2x + 3)$ (ii) $y = 2x^2(x - 1)^2$

(iii) $y = x^3(3x + 1)^2$

⑤ The sum of two numbers x and y is 13 and their product P is 40.

(i) Write down an expression for y in terms of x.

(ii) Write down an expression for P in terms of x.

(iii) Write down expressions for $\frac{dy}{dx}$ and $\frac{dP}{dx}$.

(iv) Write down the rate of change of $\frac{dP}{dx}$.

⑥ For the curve $y = 3x^3 - 2x^2 - 6x - 4$

(i) write down expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(ii) work out the gradient of the curve at the points $(-1, -7)$, $(1, -9)$ and $(2, 0)$.

(iii) work out the rate of change of the gradient at each of these points.

⑦ A formula which you will meet in Mechanics or Physics is $s = ut + \frac{1}{2}at^2$, where the letters in this case are t = time, u is the initial velocity (which will be a constant, or zero if starting from rest), a is the acceleration (which must also be constant, for this formula) and s is the distance travelled. The only variables in the formula are s and t. Using this formula $\frac{ds}{dt}$ will give the velocity after a time t has elapsed.

(i) Work out $\frac{ds}{dt}$ and hence the velocity after 12 seconds when the distance is measured in metres and time in seconds.

(ii) Work out $\frac{d^2s}{dt^2}$.

Fluency Practice

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when:

(a) $y = x^2 + 4x - 3$

(b) $y = 5x^3 + x^2 + 8x - 3$

(c) $y = x^4 - 7x^2$

(d) $y = x^2 - \frac{2}{x}$

Find the coordinates of the stationary points on each of these curves. By differentiating for a second time, establish whether these points are maximums or minimums.

(a) $y = 4x^2 - 8x$

(b) $y = 5 + 2x - x^2$

(c) $y = (8 + x)(2 - x)$

(d) $y = x^4 - 8x^2$

(e) $y = 2x^3 - 3x^2 - 12x + 5$

(f) $y = x + \frac{1}{x}$

Fluency Practice

(a) Find the gradient of the curve $y = x^2 - 3x + 7$ at the point $(3, 7)$

(b) Find the gradient of the curve $y = x^3 + 4x^2 - 9x$ at the point $(2, 6)$

(c) Find the gradient of the curve $y = x + \frac{9}{x}$ at the point $(3, 6)$

(a) Find the coordinates of the minimum point on the curve $y = x^2 - 4$

(b) Find the coordinates of the minimum point on the curve $y = x^2 + 8x + 15$

(c) Find the coordinates of the maximum point on the curve $y = 7 - 6x - x^2$

(d) Find the coordinates of the maximum point on the curve $y = 2 + 5x - x^2$

(a) Find the coordinates of the stationary points on the curve $y = x^3 - 3x^2 + 4$. By sketching the graph, determine whether each point is a minimum point or a maximum point.

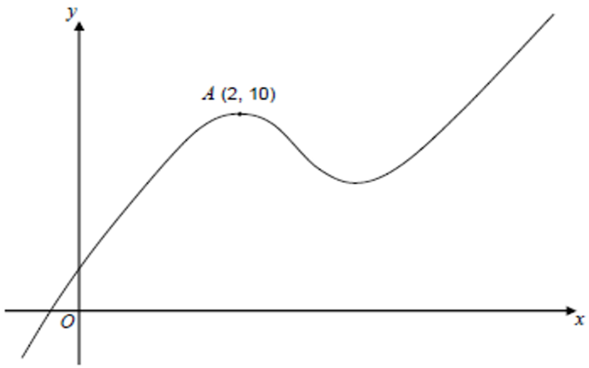
(b) Find the coordinates of the stationary point on the curve $y = 3x + \frac{12}{x^2}$. Is this point a minimum point or a maximum point?

(a) The curve with equation $y = x^2 + ax + b$ has a stationary point at $(-4, -11)$. Find the values of a and b.

(b) The curve with equation $y = c + dx - x^2$ has a stationary point at $(3, 10)$. Find the values of c and d.

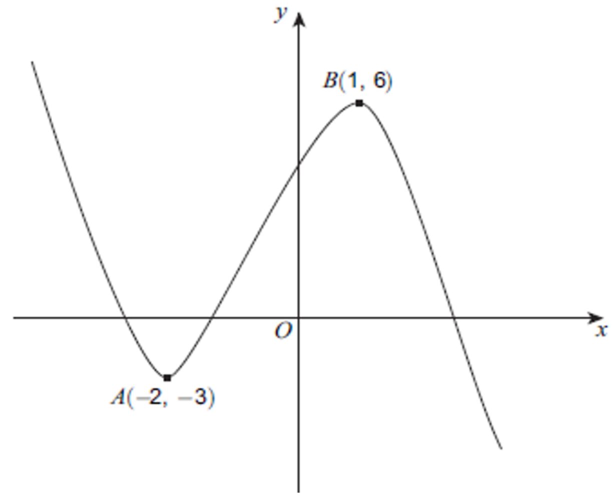
Fluency Practice

1 [Set 4 Paper 2 Q22] A sketch of $y = f(x)$, where $f(x)$ is a cubic function, is shown.



There is a maximum point at $A(2,10)$.
(a) Write down the equation of the tangent to the curve at A. ?
(b) Write down the equation of the normal to the curve at A. ?
(c) Circle the word that describes the cubic function when $x < 2$.
positive negative increasing decreasing
?

2 [June 2013 Paper 2 Q8] A sketch of $y = f(x)$ is shown. There are stationary points at A and B.



(a) Write down the equation of the tangent to the curve at A. ?
(b) Write down the equation of the normal to the curve at B. ?
(c) Circle the range of values of x for which $f(x)$ is an increasing function.
$x < -2$, $-2 < x < 1$, $-3 < x < 6$
$x > 1$
?

Fluency Practice

3 [Set 2 Paper 2 Q12] A curve has equation

$$y = x^3 - 9x^2 + 24x - 16$$

(a) Work out $\frac{dy}{dx}$?

(b) Work out the coordinates of the two stationary points on the curve.

?

4 [Set 1 Paper 2 Q14] (a) Work out the stationary points on the curve $y = x^3 - 12x$.

(b) Sketch the curve $y = x^3 - 12x$

?

6 [Specimen Paper 1 Q13] (a) Work out the coordinates of the stationary point for the curve $y = x^2 + 3x + 4$.

(b) Explain why $x^2 + 3x + 4 = 0$ has no real solutions.

?

Purposeful Practice

(a) A rectangle has a width x cm and a length $(30 - 2x)$ cm. Using calculus, find the maximum area of the rectangle.

(b) A car sales company sells x cars per week. Its revenue R per week is given by the equation $R = 0.2x^2 - 10x + 1750$. Using differentiation, find the number of cars which generates the maximum revenue, and the value of this revenue.

(a) The cost C of a car journey when driving at a speed of x mph is given by $C = \frac{720}{x} + 0.2x + 6$. Using differentiation, find the value of x that minimises the cost, and the minimum value of C.

(b) The volume of a box is given by $V = x(5 - x)^2$. Use calculus to find the maximum volume of the box, and the value of x for which this occurs.

(a) A picture frame has a perimeter of 120 cm. If the width of the frame is x cm, then show that the height of the frame is $(60 - x)$ cm. Hence use calculus to find the value of x that gives a maximum area for the frame. Calculate this maximum area.

(b) A farmer has enough stone for 80 m of dry stone walling. He wants to create a field with the largest area possible. Find the dimensions of the field that gives this maximum area.

Purposeful Practice

If you have access to a graphic calculator you will find it helpful to use it to check your answers.

① For each of the curves given below

(a) work out $\frac{dy}{dx}$ and the value(s) of x for which $\frac{dy}{dx} = 0$

(b) work out the value(s) of $\frac{d^2y}{dx^2}$ at those points

(c) classify the point(s) on the curve with these x-values

(d) work out the corresponding y-value(s)

(e) sketch the curve.

(i) $y = 1 + x - 2x^2$ (ii) $y = 12x + 3x^2 - 2x^3$

(iii) $y = x^3 - 4x^2 + 9$ (iv) $y = x(x - 1)^2$

(v) $y = x^2(x - 1)^2$ (vi) $y = x^3 - 48x$

(vii) $y = x^3 + 6x^2 - 36x + 25$ (viii) $y = 2x^3 - 15x^2 + 24x + 8$

② The graph of $y = px + qx^2$ passes through the point $(3, -15)$ and its gradient at that point is -14

(i) Work out the values of p and q.

(ii) Calculate the maximum value of y and state the value of x at which it occurs.

③ (i) Identify the stationary points of the function $f(x) = x^2(3x^2 - 2x - 3)$ and distinguish between them.

(ii) Sketch the curve $y = f(x)$.

④ The curve $y = ax^2 + bx + c$ crosses the y-axis at the point $(0, 2)$ and has a minimum point at $(3, 1)$.

(i) Work out the equation of the curve.

(ii) Check that the stationary point is a minimum.

⑤ The sum of two positive numbers p and q is 12

(i) Write q in terms of p.

(ii) S is the sum of the squares of the two numbers. Write down an expression for S in terms of p.

(iii) Work out the least value of S, checking that it is a minimum.

⑥ The sum of two positive numbers a and b is 40

(i) Write $2ab$ in terms of a.

(ii) Work out values of a and b when $2ab$ is a maximum, checking that it is a maximum.

(iii) Work out the maximum value of $2ab$.

⑦ x and y are two positive numbers whose sum is 10

(i) Express $P = xy^2$ in terms of x.

(ii) Work out the values of x for which $\frac{dP}{dx} = 0$

(iii) Use the second derivative test to identify which one gives the maximum value of P.

(iv) Comment on the implication of the other value of x that you calculated.

⑧ Netty and Mackenzie are going to climb a mountain and the equation of their path is given by $10y = x + 4x^2 - x^3$ for $x \geqslant 0$. Distances horizontally and vertically are measured in units of 1000 metres. Give all answers to 3 significant figures.

(i) How far away, horizontally, is the summit?

(ii) How much higher is the summit than where they are now?

⑨ The base of a cuboid is x cm by x cm and its height is y cm. Its volume is 216 cm^3.

(i) Write down an expression for the surface area in terms of x.

(ii) Work out the dimensions that give the minimum surface area, proving that this is a minimum.

Optimisation

One of the most useful applications of differentiation is optimisation.

It allows us to exactly calculate the ideal size for a tin or a box to minimise surface area, and therefore minimise the cost of production.

Dimensions

Baked Beans:
$Radius = 3.8cm$
$Height = 11cm$

Tuna:
$Radius = 4.3cm$
$Height = 3.5cm$

1. Calculate the volume and surface area of the baked beans tin, to 2 significant figures.
Hint: consider the net of a cylinder.

2. Find an expression for the surface area of a cylinder of radius r and volume $500cm^3$.
Hint: start by writing the height in terms of r.

3. Use differentiation to find the optimal radius and the resulting minimum surface area.
Hint: stationary points occur when the derivative equals 0.

4. Calculate the potential saving in steel if tuna were to be sold in optimal cylindrical tins (of the same volume) rather than using current dimensions.
Hint: use scale factors to generalise the result you've already found.

Extension:
The majority of tins are cylindrical in shape – an optimal circular prism (cylinder with equal diameter and height) has a surface area almost 8% smaller than that of an optimal rectangular-based prism (cube). However, even among cylindrical packaging, some are far from optimal (eg Pringles tubes, spice jars, olive oil bottles). Why is this?

2 Matrices (L2FM Only)

Fluency Practice

Write down the order of these matrices.

(a) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 3 \\ 2 & -2 \\ 4 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 2 & 6 \end{pmatrix}$

(d) $\begin{pmatrix} 0.5 & 0 \\ 1.5 & 1 \end{pmatrix}$

Work out:

(a) $\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

(b) $\begin{pmatrix} -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

(c) $\begin{pmatrix} -1 & 0 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 2 & -3 \\ 0 & -2 \end{pmatrix}$

(d) $\begin{pmatrix} 5 & 0.5 \\ -0.5 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ -4 & 0.5 \end{pmatrix}$

Work out:

(a) $2 \times \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$

(b) $4 \times \begin{pmatrix} 3 & -1 \\ 0.5 & 6 \end{pmatrix}$

(c) $-3 \times \begin{pmatrix} 11 \\ -8 \end{pmatrix}$

(d) $\frac{1}{2} \times \begin{pmatrix} 4 & -6 \\ 2 & 0 \\ 3 & 8 \end{pmatrix}$

Given that

$A = \begin{pmatrix} -2 & 3 \\ 0 & 5 \end{pmatrix}$ $B = \begin{pmatrix} 4 & -1 \\ -3 & 7 \end{pmatrix}$ $C = \begin{pmatrix} -2 & 0 \\ 8 & -3 \end{pmatrix}$

Find:

(a) $A + B - C$

(b) $C - B$

(c) $2A$

(d) $A + 2B$

(e) $3C + B$

(f) $4B - A$

(g) $2A + 3C$

(h) $-4B + A + 2B$

Fluency Practice

Is it possible to multiply the matrices shown?

(a) $\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \times \begin{pmatrix} 5 & -1 \\ 4 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} -7 & 4 \end{pmatrix} \times \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 \\ 4 & -3 \end{pmatrix} \times \begin{pmatrix} 3 & 2 & 5 \\ 6 & 0 & -1 \end{pmatrix}$

Work out:

(a) $\begin{pmatrix} 4 \\ 2 \end{pmatrix} \times \begin{pmatrix} -2 & 5 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 3 \\ 2 & 5 \end{pmatrix} \times \begin{pmatrix} -1 & 3 \\ 0 & 6 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 7 & -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & -2 \\ 3 & 7 \end{pmatrix} \times \begin{pmatrix} -1 & 4 \\ 0 & -2 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 2 \\ -5 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 6 \\ -3 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} -2 & 1 \\ 8 & 0 \end{pmatrix} \times \begin{pmatrix} -3 & 5 \\ 1 & 2 \end{pmatrix}$

(a) Given that

$\begin{pmatrix} -2 & a \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 22 \\ 9 \end{pmatrix}$

work out the value of a.

(b) Matrix **P** = $\begin{pmatrix} 2 & 3 \\ a & b \end{pmatrix}$

Matrix **Q** = $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

You are given that **PQ** = **QP**. Work out the values of a and b.

Purposeful Practice

① $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}$

$\mathbf{C} = \begin{bmatrix} 6 & -2 \\ -3 & -1 \end{bmatrix}$ $\mathbf{D} = \begin{bmatrix} 0 & 0 \\ -3 & -5 \end{bmatrix}$

$\mathbf{E} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\mathbf{F} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$\mathbf{G} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ $\mathbf{H} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

Work out

(i) **4A** (ii) **2D** (iii) **AF**
(iv) **CE** (v) **DH** (vi) **BH**
(vii) **AB** (viii) **BA** (ix) **BC**
(x) **CB** (xi) **DA** (xii) **BD**
(xiii) **AC** (xiv) **DC**

② Work out the value of p in each of the following.

(i) $\begin{bmatrix} 4 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & -1 \\ p & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$

(iii) $\begin{bmatrix} p & 1 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ p \end{bmatrix} = \begin{bmatrix} 2 \\ 17 \end{bmatrix}$

(iv) $\begin{bmatrix} p & 4p \\ p & -2p \end{bmatrix}\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ p & 4 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 16 & 9 \end{bmatrix}$

(vi) $\begin{bmatrix} 4 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2p \\ -1 & p \end{bmatrix} = \begin{bmatrix} 9 & -14 \\ 0 & 0 \end{bmatrix}$

③ Work out the values of x and y in each of the following.

(i) $\begin{bmatrix} 2 & 1 \\ 1 & y \end{bmatrix}\begin{bmatrix} x \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & x \\ 2y & 3y \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \end{bmatrix}$

(iii) $\begin{bmatrix} -3 & 0 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ x & y \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ -4 & 10 \end{bmatrix}$

(iv) $\begin{bmatrix} x & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ x & y \end{bmatrix} = \begin{bmatrix} -9 & -5 \\ -2 & -4 \end{bmatrix}$

④ Given that $\begin{bmatrix} 5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

(i) write down two equations in x and y

(ii) work out x and y by solving the pair of simultaneous equations.

⑤ Work out the values of a and b in each of the following.

(i) $\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 21 \end{bmatrix}$

(ii) $\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = 6\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & b \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & 2a \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 11 \\ 2 & -2 \end{bmatrix}$

(iv) $\begin{bmatrix} 5 & 1 \\ 3a & b+1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 12 & 3 \end{bmatrix}$

PS ⑥ Given that $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, work out the values of a, b, c and d.

PS ⑦ $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 0 & k \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$. Work out the value of k such that **AB** = **BA**.

PS ⑧ Given that $\begin{bmatrix} p & 4 \\ 1 & q \end{bmatrix}\begin{bmatrix} r & p-4 \\ q+8 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 3p+4 \\ -17 & -3p \end{bmatrix}$, work out the values of p, q and r.

PS ⑨ (i) Calculate the product $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(ii) Hence write down the matrix **M**, in terms of a, b, c and d, such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Discussion point

Are there any values of a, b, c and d, for which the matrix **M** in question 9 does not exist?

Purposeful Practice

$A = \begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix} \quad B = \begin{pmatrix} -2 & 0 \\ 4 & 5 \end{pmatrix}$

(a) Given that $B + C = I$, find C

(b) Given that $D - A = I$, find D

(c) Given that $B + 2I = E$, find E

(a) Given that

$$\begin{pmatrix} x & -2 \\ -7 & y \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} = I$$

Find the values of x and y.

(b) Given that

$$\begin{pmatrix} 4 & -1 \\ -7 & 2 \end{pmatrix}\begin{pmatrix} 2 & p \\ q & 4 \end{pmatrix} = I$$

Find the values of p and q.

(a) Find I^2

(b) Given that $2A + I^2 = \begin{pmatrix} 6 & -4 \\ -1 & 5 \end{pmatrix}$

find A.

(a) Given that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix} = I$

find the values of a, b, c and d.

(b) Given that $\begin{pmatrix} -5 & 3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = I$

find the values of a, b, c and d.

(c) Given that

$$\begin{pmatrix} x & \frac{1}{2} \\ -2 & y \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{z}{4} \\ -\frac{2}{3} & -\frac{4}{9} \end{pmatrix} = I^2$$

find the values of $x, y,$ and z.

Fluency Practice

1 Work out

(a) $\begin{pmatrix} 4 & 2 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix}$?

(b) $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -3 \\ -4 \end{pmatrix}$?

(c) $2\begin{pmatrix} 5 & -2 \\ 6 & -3 \end{pmatrix}$?

(d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$?

(e) $6\begin{pmatrix} -4 & 7 \\ -1 & -3 \end{pmatrix}$?

(f) $\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ 6 \end{pmatrix}$?

2 $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$

Work out

(a) **AB** ?

(b) **BC** ?

(c) 3**A** ?

(d) **BA** ?

(e) –**C** ?

(f) $\mathbf{B}\begin{pmatrix} 1 & -4 \\ -5 & 7 \end{pmatrix}$?

3 $\mathbf{P} = \begin{pmatrix} -2 & 0 \\ 5 & 1 \end{pmatrix}$ $\mathbf{Q} = \begin{pmatrix} -4 & 1 \\ 3 & -2 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Work out

(a) $\mathbf{P}^2$?

(b) **QP** ?

(c) 5**Q** ?

(d) **PC** ?

(e) **IQ** ?

(f) 3**I** ?

Fluency Practice

4

(a) $\begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}\begin{pmatrix}0 & 3\\1 & -4\end{pmatrix}$?

(b) $\begin{pmatrix}-3 & -2\\-1 & 5\end{pmatrix}\begin{pmatrix}-2 & 4\\3 & 4\end{pmatrix}$?

(c) $\begin{pmatrix}3 & 2\\7 & 5\end{pmatrix}\begin{pmatrix}5 & -2\\-7 & 3\end{pmatrix}$?

(d) $\begin{pmatrix}10 & -7\\9 & 8\end{pmatrix}\begin{pmatrix}2 & 4\\-2 & 3\end{pmatrix}$?

(e) $\begin{pmatrix}1 & -2\\3 & -5\end{pmatrix}\begin{pmatrix}2 & 3\\1 & 4\end{pmatrix}$?

(f) $\begin{pmatrix}2 & 3\\1 & 4\end{pmatrix}\begin{pmatrix}1 & -2\\3 & -5\end{pmatrix}$?

5 Work out, giving your answers as simply as possible.

(a) $\begin{pmatrix}\sqrt{2} & 1\\-1 & 3\sqrt{2}\end{pmatrix}\begin{pmatrix}\sqrt{2} & 0\\-3 & -2\sqrt{2}\end{pmatrix}$?

(b) $\begin{pmatrix}-\frac{1}{2} & -1\\\frac{3}{2} & 5\end{pmatrix}\begin{pmatrix}-2 & 4\\-\frac{1}{2} & 3\end{pmatrix}$?

(c) $\begin{pmatrix}3 & 2\\7 & 5\end{pmatrix}^2$?

(d) $\begin{pmatrix}3\sqrt{3} & -4\\2 & 3\sqrt{3}\end{pmatrix}\begin{pmatrix}\sqrt{3} & 1\\-4 & 0\end{pmatrix}$?

(e) $\begin{pmatrix}\frac{1}{3} & \frac{1}{2}\\\frac{2}{3} & \frac{1}{4}\end{pmatrix}\begin{pmatrix}2 & 3\\1 & 4\end{pmatrix}$?

(f) $\begin{pmatrix}\sqrt{2} & 2\\7 & \sqrt{3}\end{pmatrix}^2$?

6 Work out, giving your answers as simply as possible.

(a) $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}p\\p+1\end{pmatrix}$?

(b) $\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$?

(c) $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}m\\2m\end{pmatrix}$?

(d) $\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}-a & 0\\0 & a\end{pmatrix}$?

(e) $\begin{pmatrix}4t & 0\\0 & 4t\end{pmatrix}\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}$?

(f) $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix}$?

Fluency Practice

7

Work out, giving your answers as simply as possible.

(a) $\begin{pmatrix} 2x & -3 \\ -5 & 4x \end{pmatrix}\begin{pmatrix} x & 3x \\ -3 & 0 \end{pmatrix}$?

(b) $\begin{pmatrix} a & 3a \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 7 & 8 \\ -10 & 11 \end{pmatrix}$?

(c) $\begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix}^2$?

(d) $\begin{pmatrix} y & y \\ -3 & x \end{pmatrix}\begin{pmatrix} 2 & 3y \\ 0 & 1 \end{pmatrix}$?

(e) $\begin{pmatrix} a+1 & a \\ a+2 & a+1 \end{pmatrix}\begin{pmatrix} a+1 & -a \\ -a-2 & a+1 \end{pmatrix}$?

(f) $\begin{pmatrix} 3x & -3 \\ -9 & x+1 \end{pmatrix}^2$?

Fluency Practice

1

$$\begin{pmatrix} -2 & a \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 22 \\ 9 \end{pmatrix}$$

?

Work out the value of a.

2 June 2013 Paper 2 Q11

(a) Work out $\begin{pmatrix} 2 & -1 \\ \frac{1}{3} & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ a & c \end{pmatrix}$

?

Give your answer in terms of a, b and c.

(b) You are given that $\begin{pmatrix} 2 & -1 \\ \frac{1}{3} & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ a & c \end{pmatrix} = \mathbf{I}$ where **I** is the identity matrix.

Work out the values of a, b and c.

?

3 Set 2 Paper 2 Q16

Matrix $\mathbf{P} = \begin{pmatrix} 2 & 3 \\ a & b \end{pmatrix}$ Matrix $\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

You are given that $\mathbf{PQ} = \mathbf{QP}$

Work out the values of a and b.

?

4 Set 4 Paper 1 Q17

$$\begin{pmatrix} 2 & a \\ 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Work out **all** possible pairs of values of a and b.

?

Problem Solving

$$PQ = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$QP = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$$

2×2 matrix multiplication is **not** necessarily commutative

BUT, can you find 2×2 matrices A and B such that AB = BA? (it is possible!)

Purposeful Practice

① Work out the image of point (4, 2) for the transformation defined by matrix $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

② Work out the image of point (1, −3) for the transformation represented by matrix $\begin{bmatrix} 0 & -3 \\ -1 & 5 \end{bmatrix}$.

③ Work out the image of point (−2, −3) for the transformation defined by matrix $\begin{bmatrix} -2 & -3 \\ 2 & -1 \end{bmatrix}$.

④ The image of point (4, 3) under the transformation matrix $\begin{bmatrix} 2 & 1 \\ c & 3 \end{bmatrix}$ is (11, 1). Work out the value of c.

⑤ The image of point $(a, 1)$ under the transformation matrix $\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$ is (7, 17). Work out the value of a.

⑥ The image of point (3, −2) under the transformation matrix $\begin{bmatrix} a & 2a \\ b & 3 \end{bmatrix}$ is (b, b). Work out the values of a and b.

⑦ The transformation matrix $\begin{bmatrix} 2c & d \\ c & -d \end{bmatrix}$ maps the point (2, 5) to the point (−6, 12). Work out the values of c and d.

⑧ Given that $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, show that $\mathbf{A}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\mathbf{A}^2$ means $\mathbf{A} \times \mathbf{A}$, i.e. $\mathbf{AA}$

⑨ Given that $\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 7 & p \end{bmatrix} = \mathbf{I}$, work out the value of p.

PS ⑩ Under the transformation matrix $\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, point D is mapped to (11, 10). Work out the coordinates of D.

PS ⑪ The matrices $\mathbf{M} = \begin{bmatrix} a & 3 \\ 2 & b \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} a+1 & b+2 \\ -a & 2 \end{bmatrix}$ satisfy the equation $\mathbf{MN} = c\mathbf{I}$. Work out the values of a, b and c.

PS ⑫ (i) Under the transformation matrix $\begin{bmatrix} 4 & -1 \\ 6 & -1 \end{bmatrix}$, which of the following points is **not** invariant?

(1, 3) (2, 4) (3, 9) (5, 15)

(ii) If the point (x, y) is invariant under $\begin{bmatrix} 4 & -1 \\ 6 & -1 \end{bmatrix}$, identify a condition for all of the invariant points of this transformation.

Note

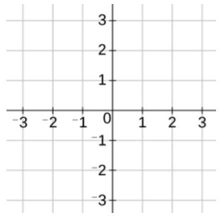
Question 12 is for interest only as this specification does not assess a candidate's knowledge of invariant points, i.e. a point which does not move (see GCSE Maths).

Purposeful Practice

Rotations Using Matrices

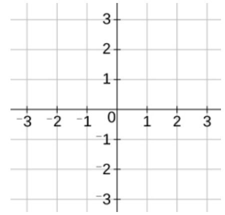
(a)

By considering the unit square, determine the matrix which describes a rotation $90°$ clockwise about the origin.



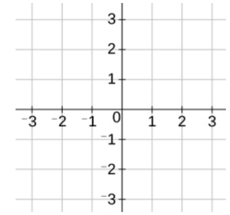
(b)

Describe fully the single transformation represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



(c)

By considering the unit square, determine the matrix which describes a rotation $180°$ about the origin.



(d)

The point $(1, -6)$ is mapped onto the point (a, b) when rotated $90°$ anti-clockwise about the origin. Using matrix algebra, find the values of a and b.

(e)

The point (c, d) is mapped onto the point $(2, 4)$ when rotated $270°$ anti-clockwise about the origin. Using matrix algebra, find the values of c and d.

(f)

A triangle with vertices at $(1, 1)$, $(5, 2)$ and $(4, -1)$ is rotated $180°$ about the origin. Use matrix algebra to find the coordinates of the vertices of the rotated triangle.

(g)

Use matrix algebra to show that a rotation of $90°$ clockwise about the origin, followed by a rotation of $180°$ is equivalent to a rotation of $90°$ anti-clockwise about the origin.

(h)

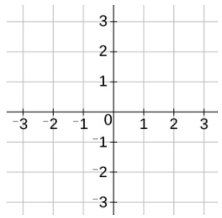
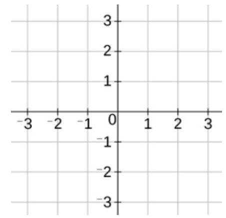
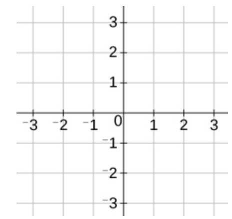
The point $(a, 6)$ is mapped onto the point $(b, -4)$ following a rotation of $90°$ anti-clockwise about the origin. Use matrix algebra to find the values of a and b.

(i)

The point $(x, 2y + 6)$ is mapped onto the point $(2x, y - 7)$ following a rotation of $90°$ clockwise about $(0, 0)$. Use matrix algebra to find the values of x and y.

Purposeful Practice

Reflections Using Matrices

(a)	(b)	(c)
By considering the unit square, determine the matrix which describes a reflection in the x-axis. 	Describe fully the single transformation represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 	By considering the unit square, determine the matrix which describes a reflection in the line $y = -x$. 
(d)	**(e)**	**(f)**
The point $(-4, 2)$ is mapped onto the point (a, b) when reflected in the x-axis. Using matrix algebra, find the coordinates (a, b).	The point (c, d) is mapped onto the point $(7, -5)$ when reflected in the line $y = -x$. Using matrix algebra, find the coordinates (c, d).	A triangle with vertices at $(0, 5)$, $(4, 3)$ and $(1, -1)$ is reflected in the line $y = x$. Use matrix algebra to find the coordinates of the vertices of the reflected triangle.
(g)	**(h)**	**(i)**
A triangle with vertices at $(0, 1)$, $(1, 0)$ and $(3, 2)$ is reflected so its vertices map to $(0, -1)$, $(-1, 0)$ and $(-2, -3)$. Find the transformation matrix and the line of reflection.	The point $(-2, a)$ is mapped onto the point $(b, 3)$ following a reflection in the line $x = 0$. Use matrix algebra to find the values of a and b.	The point $(x, 3x - 7)$ is mapped onto the point $(y + 3, y)$ following a reflection in the line y-axis. Use matrix algebra to find the values of x and y.

Purposeful Practice

A triangle with coordinates (3,2), (5,2) and (3,6) is transformed by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. By pre-multiplying, find the coordinates of the transformed triangle. Draw this transformation on a grid and hence describe it fully.

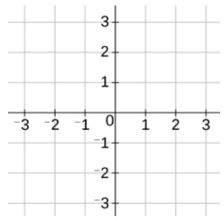
A triangle with coordinates (-3,2), (-5,2) and (-3,5) is transformed by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. By pre-multiplying, find the coordinates of the transformed triangle. Draw this transformation on a grid and hence describe it fully.

A triangle with coordinates (2,3), (4,3) and (4,7) is transformed by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. By pre-multiplying, find the coordinates of the transformed triangle. Draw this transformation on a grid and hence describe it fully.

A triangle with coordinates (3,1), (5,1) and (3,5) is transformed by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. By pre-multiplying, find the coordinates of the transformed triangle. Draw this transformation on a grid and hence describe it fully.

The transformation matrix $\begin{pmatrix} a & 2 \\ -1 & 1 \end{pmatrix}$ maps the point $(3,\ 4)$ onto the point $(2,\ b)$. Work out the values of a and b.

Purposeful Practice

Enlargements Using Matrices		
(a)	**(b)**	**(c)**
By considering the unit square, determine the matrix which describes an enlargement about the origin with scale factor 3. 	Describe fully the single transformation represented by the matrix $\begin{pmatrix} \frac{5}{2} & 0 \\ 0 & \frac{5}{2} \end{pmatrix}$	Use matrix algebra to show that an enlargement of scale factor 2 about $(0, 0)$, followed by an enlargement of scale factor 1.5 about $(0, 0)$ is equivalent to an enlargement of scale factor 3 about $(0, 0)$.
(d)	**(e)**	**(f)**
The point $(-5, 3)$ is mapped onto the point (a, b) when enlarged by a scale factor 2 about the origin. Using matrix algebra, find the values of a and b.	The unit square OABC with coordinates O(0, 0), A(0, 1), B(1, 1) and C(1, 0) is mapped to OA′B′C′ under matrix $\begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}$. Use matrix algebra to find the coordinates of A′, B′ and C′.	The point (c, d) is mapped onto the point $(-1, -4)$ when enlarged by a scale factor 0.5 about the origin. Using matrix algebra, find the values of c and d.
(g)	**(h)**	**(i)**
Use matrix algebra to show that an enlargement of scale factor 2 about $(0, 0)$, followed by an enlargement of scale factor -0.5 about $(0, 0)$ is the same as a rotation of $180°$ about the origin.	The point $(a, 3)$ is mapped to the point $(6, 2a)$ when enlarged with scale factor b about the origin. Use matrix algebra to find the possible values of a and b.	The point $(x - 4,\ y)$ is mapped to the point $(2y,\ 2x - 18.5)$ when transformed under the matrix $\begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}$. Find the values of x and y.

Purposeful Practice

① Work out the 2×2 matrix that represents each of the following transformations.

(i) Reflection in the x-axis.

(ii) Rotation of 90° about O.

(iii) Enlargement, scale factor 2, centre the origin.

(iv) Reflection in the y-axis.

(v) Reflection in the line $y = x$.

(vi) Rotation by 180°, centre the origin.

(vii) Reflection in the line $y = -x$.

(viii) Enlargement, scale factor −3, centre O.

(ix) Enlargement, centre O, scale factor $\frac{1}{2}$.

② The unit square OABC is transformed by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to OA′B′C′. Show the image on a diagram, labelling each vertex.

③ The unit square OABC is transformed by the matrix $\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ to OA′B′C′.

Show the image on a diagram, labelling each vertex.

④ Describe fully the transformations given by the following matrices.

(i) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(vii) $\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$ (viii) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Note

Question 8 is for interest only as this specification only includes rotations of 90°, 180° and 270°.

⑤ The unit square OABC is transformed to OA′B′C′.
OA′B′C′ is shown on the diagram.

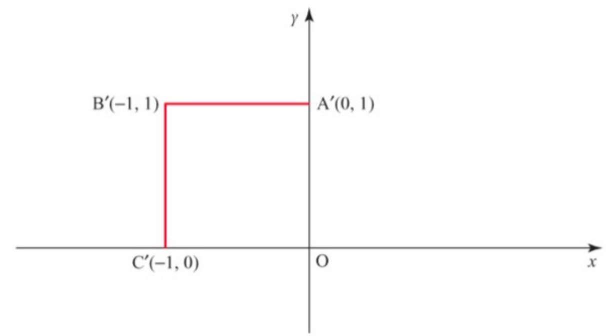


Figure 9.5

Work out the matrix for the transformation.

⑥ The unit square OABC is transformed by the matrix $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ to OA′B′C′.

Work out the area of OA′B′C′.

⑦ The unit square OABC is transformed by the matrix $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ to OA′B′C′.

The area of OA′B′C′ is 64 square units.

Work out the two possible values of k.

⑧ (i) Draw a diagram to show the unit square OABC rotated 45° about the origin.

(ii) Work out the coordinates of A′ and C′ (the images of A and C).

(Hint: $\sin 45° = \frac{\sqrt{2}}{2}$ and $\cos 45° = \frac{\sqrt{2}}{2}$.)

(iii) Hence write down the transformation matrix for a rotation of 45° about the origin.

Purposeful Practice

1 [Jan 2013 Paper 2 Q15] Describe fully the **single** transformation represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

?

2 [Set 2 Paper 1 Q4] The transformation matrix $\begin{pmatrix} a & 2 \\ -1 & 1 \end{pmatrix}$ maps the point $(3,4)$ onto the point $(2, b)$. Work out the values of a and b.

?

3 [Set 3 Paper 1 Q6] The matrix $\begin{pmatrix} a & b \\ -a & 2b \end{pmatrix}$ maps the point $(5,4)$ onto the point $(1,17)$. Work out the values of a and b.

?

4 [Worksheet 2 Q5] Work out the image of the point *D* (–1, 2) after transformation by the matrix $\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$

?

5 [Worksheet 2 Q6] The point $A(m, n)$ is transformed to the point A' (–2, 0) by the matrix $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$

Work out the values of *m* and *n*.

?

6 [Worksheet 2 Q8] Describe fully the transformation given by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

?

7 [Worksheet 2 Q9] The unit square *OABC* is transformed by the matrix $\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$ to the square $OA'B'C'$.

The area of $OA'B'C'$ is 27. Work out the exact value of *h*.

?

Purposeful Practice

Harder Transformations Using Matrices

(a)	(b)	(c)
Find the single matrix that represents an enlargement about the origin with scale factor 3, followed by a rotation of $90°$ clockwise about the origin.	Find the single matrix that represents a reflection in the y-axis, followed by a rotation of $180°$ about the origin.	$P = \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix}$ $Q = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$ Matrices P and Q represent different transformations. Find the single matrix that represents transformation P followed by transformation Q.
(d)	**(e)**	**(f)**
The point P $(4, -2)$ is mapped to the point Q following a reflection in the line $y = x$, then an enlargement with scale factor 2 about the origin. Use matrix algebra to find the coordinates of point Q.	The point (a, b) is mapped to the point $(-5, 1)$ following a rotation of $180°$ about the origin, then a reflection in the x-axis. Using matrix algebra, find the coordinates (a, b).	The matrix $\begin{pmatrix} 0 & b \\ -2 & 4 \end{pmatrix}$ maps the point $(a, -3)$ onto the point $(-9, 5)$. Use matrix algebra to find the values of a and b.
(g)	**(h)**	**(i)**
The transformation matrix $\begin{pmatrix} a & 2b \\ -a & 3 \end{pmatrix}$ maps the point $(2, -1)$ to the point $(6, 7)$. Find the values of a and b.	The transformation matrix $\begin{pmatrix} b & 2a \\ a & -b \end{pmatrix}$ maps the point $(6, 3)$ to the point $(24, b)$. Find the values of a and b.	Point $(c, 4)$ is mapped to the point $(-2, d)$ by the transformation matrix $\begin{pmatrix} c & -3 \\ 2 & -1 \end{pmatrix}$. Use matrix algebra to find the two possible values of c and d.

Purposeful Practice

Find the matrices that represent the following transformations:

(a) A reflection in the x-axis, followed by a rotation through $180°$ centre the origin.

(b) An enlargement with centre the origin and scale factor 2, followed by a reflection in the line $y = x$.

(c) A reflection in the y-axis followed by a reflection in the line $y = x$.

(d) A reflection in the line $y = x$ followed by enlargement about the origin with scale factor 3.

Point $(3, -2)$ is transformed by the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ followed by a further transformation by the matrix $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

(i) Work out the matrix for the combined transformation.

(ii) Work out the co-ordinates of the image point of P.

Point $(-1,4)$ is transformed by the matrix $\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}$ followed by a further transformation by the matrix $\begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$.

(i) Work out the matrix for the combined transformation.

(ii) Work out the co-ordinates of the image point of W.

The transformation matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ maps a point P to Q. The transformation matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ maps point Q to point R. Point R is $(-4,3)$. Work out the coordinates of point P.

Purposeful Practice

① Point P(3, −2) is transformed by $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, followed by a further transformation $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$.

(i) Work out the matrix for the combined transformation.

(ii) Work out the x-coordinate of the image point of P.

② Point W(−1, 4) is transformed by $\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$, followed by a further transformation $\begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$.

(i) Work out the matrix for the combined transformation.

(ii) Work out the coordinates of the image point of W.

③ The unit square is reflected in the x-axis followed by a rotation through 180°, centre the origin.

Work out the matrix for the combined transformation.

④ The unit square is enlarged, centre the origin, scale factor 2, followed by a reflection in the line $y = x$.

Work out the matrix for the combined transformation.

⑤ The unit square is rotated by 90°, centre the origin, followed by a reflection in the y-axis.

Work out the matrix for the combined transformation.

⑥ Matrix $\mathbf{P} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(i) Describe the transformation that this matrix represents.

(ii) Work out $\mathbf{P}^2$.

(iii) Use transformations to interpret your answer to part (ii).

⑦ Matrix $\mathbf{D} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Matrix $\mathbf{E} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(i) Describe the transformation that matrix **D** represents.

(ii) Describe the transformation that matrix **E** represents.

(iii) Work out **DE**.

(iv) Describe the transformation that matrix **DE** represents.

(v) Use transformations to interpret your answer to part (iv).

(vi) Use transformations to explain why **ED** = **DE**.

PS ⑧ Use matrices to prove the following:

enlargement, centre O, scale factor 3 followed by enlargement, centre O, scale factor −2 is equivalent to a single enlargement, centre O, scale factor k, where k is an integer to be found.

PS ⑨ (i) Calculate the combined transformation matrix of $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$, followed by $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, and finally $\begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$.

(ii) Under this combined transformation, point A is mapped to the point $(-14, 7)$. Work out the coordinates of point A.

PS ⑩ Identify three matrices **A**, **B** and **C** such that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

Purposeful Practice

1 Point $(3, -2)$ is transformed by the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ followed by a further transformation by the matrix $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

(i) Work out the matrix for the combined transformation.

?

(ii) Work out the co-ordinates of the image point of P.

?

2 Point $(-1,4)$ is transformed by the matrix $\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}$ followed by a further transformation by the matrix $\begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$.

(i) Work out the matrix for the combined transformation.

?

(ii) Work out the co-ordinates of the image point of W.

?

3 The unit square is reflected in the x-axis followed by a rotation through 180° centre the origin. Work out the matrix for the combined transformation.

?

4 The unit square is enlarged, centre the origin, scale factor 2 followed by a reflection in the line $y = x$. Work out the matrix for the combined transformation.

?

Purposeful Practice

5 [Jan 2013 Paper 2 Q17] $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ represents a reflection in the y-axis. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents a reflection in the line $y = x$.
Work out the matrix that represents a reflection in the y-axis followed by a reflection in the line $y = x$.

?

6 [June 2012 Paper Q22] The transformation matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ maps a point P to Q. The transformation matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ maps point Q to point R.
Point R is $(-4,3)$. Work out the coordinates of point P.

?

7 [Set 1 Paper Q14b] The unit square OABC is transformed by reflection in the line $y = x$ followed by enlargement about the origin with scale factor 2. What is the matrix of the combined transformation?

?

$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.
The point $P(2,7)$ is transformed by matrix BA to P'. Show that P' lies on the line
$7x + 2y = 0$.

?