KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

Year 11

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

2025 Level 2 Further Mathematics 2026

Unit 27 Booklet

HGS Maths

Tasks

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

1 Binomial Expansion

Introduction

Expand the following (by directly multiplying out the brackets):

$$(a+b)^0 \equiv 1$$

$$(a+b)^1 \equiv 1a + 1b$$

$$(a+b)^2 \equiv 1a^2 + 2ab + 1b^2$$

$$(a+b)^3 \equiv 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

$$(a+b)^4 \equiv 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

Each coefficient is the sum of the two above it.

What do you notice about?

a the coefficients of the terms?

b the powers of a and b?

The power of a goes down by 1 each time (starting with the power of the original bracket, and descending to $a^0 = 1$, thus not appearing).

The power of b goes up by 1 each time (starting from 0).

Pascal's Triangle

In <u>Pascal's Triangle,</u> each term (except for the 1s) is the sum of the two terms above.

The second number of each row tells us what row we should use for an expansion.

If we were expanding $(2+x)^4$, the power is 4, so we use this row.

1 — 0th row

1 1 — 1st row

1 2 1 — 2nd row

1 3 3 1 — 3rd row

1 4 6 4 1 — 4th row

1 5 10 10 5 1 — 5th row

Tip: We <u>highly recommend</u> memorising each row up to what you see here.

We'll see later **<u>why</u>** each row gives us the coefficients in an expansion of $(a+b)^n$

Worked Example	Your Turn
Use Pascal's triangle to find the binomial expansion of $(5x-2)^4$ Give each term in its simplest form.	Use Pascal's triangle to find the binomial expansion of $(2x-5)^4$ Give each term in its simplest form.

Worked Example	Your Turn
Use Pascal's triangle to find the first three terms, in descending powers of x, of the binomial expansion of $(2x-3)^6$ Give each term in its simplest form.	Use Pascal's triangle to find the first three terms, in descending powers of x, of the binomial expansion of $(3x-2)^5$ Give each term in its simplest form.

Dr Frost 501c, 501a and 501b

Binomial Expansion

(a) Expand and simplify $(x+2)^3$

Pascal's Triangle	Powers of 1st term	Powers of 2nd term	Simplified
1	x^3	2^0	x^3
3	x^2	2^1	$6x^2$
3	x^1	2^2	
1	x^0	2^3	

$= x^3 + 6x^2 +$

(b) Expand and simplify $(x-5)^3$

Pascal's Triangle	Powers of 1st term	Powers of 2nd term	Simplified
1	x^3	$(-5)^0$	x^3
3	x^2	$(-5)^1$	$-15x^2$

$= x^3 - 15x^2 +$

(c) Expand and simplify $(x+y)^4$

Pascal's Triangle	Powers of 1st term	Powers of 2nd term	Simplified
1	x^4	y^0	
4	x^3	y^1	
6	x^2	y^2	

(d) Expand and simplify $(2x+1)^4$

Pascal's Triangle	Powers of 1st term	Powers of 2nd term	Simplified
1	$(2x)^4$	1^0	$16x^4$

(e) Expand and simplify $(3x-2)^5$

Pascal's Triangle	Powers of 1st term	Powers of 2nd term	Simplified
1	$(3x)^5$	$(-2)^0$	
5	$(3x)^4$	$(-2)^1$	

(f) Expand and simplify $(4-y)^5$

Pascal's Triangle	Powers of 1st term	Powers of 2nd term	Simplified
1	4^5	$(-y)^0$	

Factorials

There are 4 pieces of different fruit.
How many ways are there of arranging them in a line?

This is 4!

1st choice		2nd choice		3rd choice		4th choice	
4	$\times$	3	$\times$	2	$\times$	1	$= 24$

We can choose any of the pieces of fruit for the first slot.

We only have 3 remaining pieces of fruit to put in the second slot.

$n! = n\times(n-1)\times(n-2)\times\cdots\times 1$, said "$n$ factorial", is the number of ways of arranging n items in a particular order.

The Choose Function

Determine the number of ways of picking 3 pieces of 5 fruit, where the order within our selection does not matter.

We say $\binom{5}{3}$ as "5 choose 3"

$\binom{n}{r}$ (also written ${}^nC_r$) is the **choose function** and gives the number of ways of selecting r items from n **where order does not matter.**

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$\binom{5}{3} = \frac{5!}{3!\times 2!} = 10$$

You can avoid using the formula by using the choose button on your calculator.

Fluency Practice

Calculate the value of the following. You may use the factorial button, but not the nCr button.

a $5!$

b $\binom{5}{3}$

c $0!$

d $\binom{20}{1}$

e $\binom{20}{0}$

f $\binom{20}{2}$

g $\binom{20}{18}$

a	?
b	?
c	?
d	?
e	?
f	?
g	?

Using the Choose Function for Binomial Expansion

If the power in the binomial expansion is large, e.g. $(x+3)^{20}$, **it is no longer practical to go this far down Pascal's triangle**. We can instead use the choose function to get numbers from anywhere within the triangle.

1	0th row	$\binom{0}{0}$
1 1	1st row	$\binom{1}{0}$ $\binom{1}{1}$
1 2 1	2nd row	$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$
1 3 3 1	3rd row	$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$
1 4 6 4 1		$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

Deeper Thinking:

1. The top number matches the row number.
2. The bottom number goes from 0 and eventually matches the top number.
3. It's easy to see from the symmetry of Pascal's triangle that $\binom{4}{1} = \binom{4}{3}$ for example.

Why Can We Use the Choose Function?

Suppose we wanted the coefficient of the a^3b term. How could this arise from the expansion of the brackets?

To expand multiple brackets, we consider all possible combinations of **choosing 1 term from each bracket** and multiplying these terms.

$$(a+b)^4 \equiv (a+b)(a+b)(a+b)(a+b)$$

$$+\boldsymbol{a^3b} \quad +\boldsymbol{a^3b} + \cdots$$

We need to choose an a from 3 of the brackets, and the b from the remaining bracket. Here is one way.

Total of a^3b terms:

$$\binom{4}{3} a^3 b$$

More generally, if we are choosing 3 of the 4 brackets to use the a (with the b used in the remaining brackets), there must be $\binom{4}{3}$ ways of doing so.

Worked Example	Your Turn
Find the first three terms, in ascending powers of x, of the binomial expansion of $(3x-2)^8$. Give each term in its simplest form.	Find the first three terms, in ascending powers of x, of the binomial expansion of $(2x-3)^7$. Give each term in its simplest form.

Dr Frost 501e and 501d

Worked Example	Your Turn
Find the first three terms, in ascending powers of x, of the binomial expansion of $\left(3x - \frac{2}{3}\right)^7$ Give each term in its simplest form.	Find the first three terms, in ascending powers of y, of the binomial expansion of $\left(3y + \frac{3}{4}\right)^5$ Give each term in its simplest form.

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Worked Example	Your Turn
Find the first three terms, in ascending powers of x, of the binomial expansion of $(2 - gx)^7$. Give each term in its simplest form.	Find the first three terms, in ascending powers of y, of the binomial expansion of $(2 + fy)^9$. Give each term in its simplest form.

Dr Frost 501h

Worked Example	Your Turn
Find the expansion of $(2x-3y^2)^5$	Find the expansion of $(3x^2-2y)^4$

Fluency Practice

More Binomial Expansion

(a) Expand and simplify $(x-2)^6$ $1 \times x^6 \times (-2)^0$ $+6 \times x^5 \times (-2)^1$	**(b)** Expand and simplify $\left(3x+\frac{1}{2}\right)^4$ $1 \times (3x)^4 \times \left(\frac{1}{2}\right)^0$
(c) Expand and simplify $(x^2+5)^3$	**(d)** Expand and simplify $\left(\frac{x}{3}-1\right)^5$
(e) Expand and simplify $\left(\frac{3}{2}-5y\right)^4$	**(f)** Expand and simplify $\left(2x-\sqrt{3}\right)^3$

Determining a Specific Term of the Expansion

You may want a specific term of an expansion **<u>without</u>** having to list out unnecessary terms.

What is the coefficient of the x^4 term in the expansion of $(3-4x)^{10}$?

The term will be a product of a binomial coefficient, a power of 3 and a power of $(-4x)$

We want a x^4 term, so what should this exponent be?

$$\binom{10}{6} 3^6 (-4x)^4$$

Due to symmetry in Pascal's triangle, it doesn't matter whether we use either of the exponents, 4 or 6.

Therefore, what is this exponent?

$$\equiv 210 \times 729 \times 256 x^4$$

$$\equiv 39191040 x^2$$

Coefficient of x^2 is 39191040

Fluency Practice

a

$(2+3x)^7$

Wanted: x^3 term

?

b

$(1-5x)^6$

Wanted: x^2 term

?

c

$(4-x)^{20}$

Wanted: x^7 term

?

d

$(kx+a)^n$

Wanted: x^5 term

?

Worked Example	Your Turn
Find the coefficient of x^3 in the expansion of $(2-3x)^4$	Find the coefficient of x^4 in the expansion of $(3-2x)^5$

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Worked Example	Your Turn
Find the coefficient of x^{20} in $(1-3x^5)^7$	Find the coefficient of x^{20} in $(1-2x^4)^7$

Worked Example	Your Turn
The coefficient of x^3 in the expansion of $(3-cx)^5$ is 720. Find the possible value(s) of the constant c.	The coefficient of x^3 in the expansion of $(2-cx)^5$ is -1080. Find the possible value(s) of the constant c.

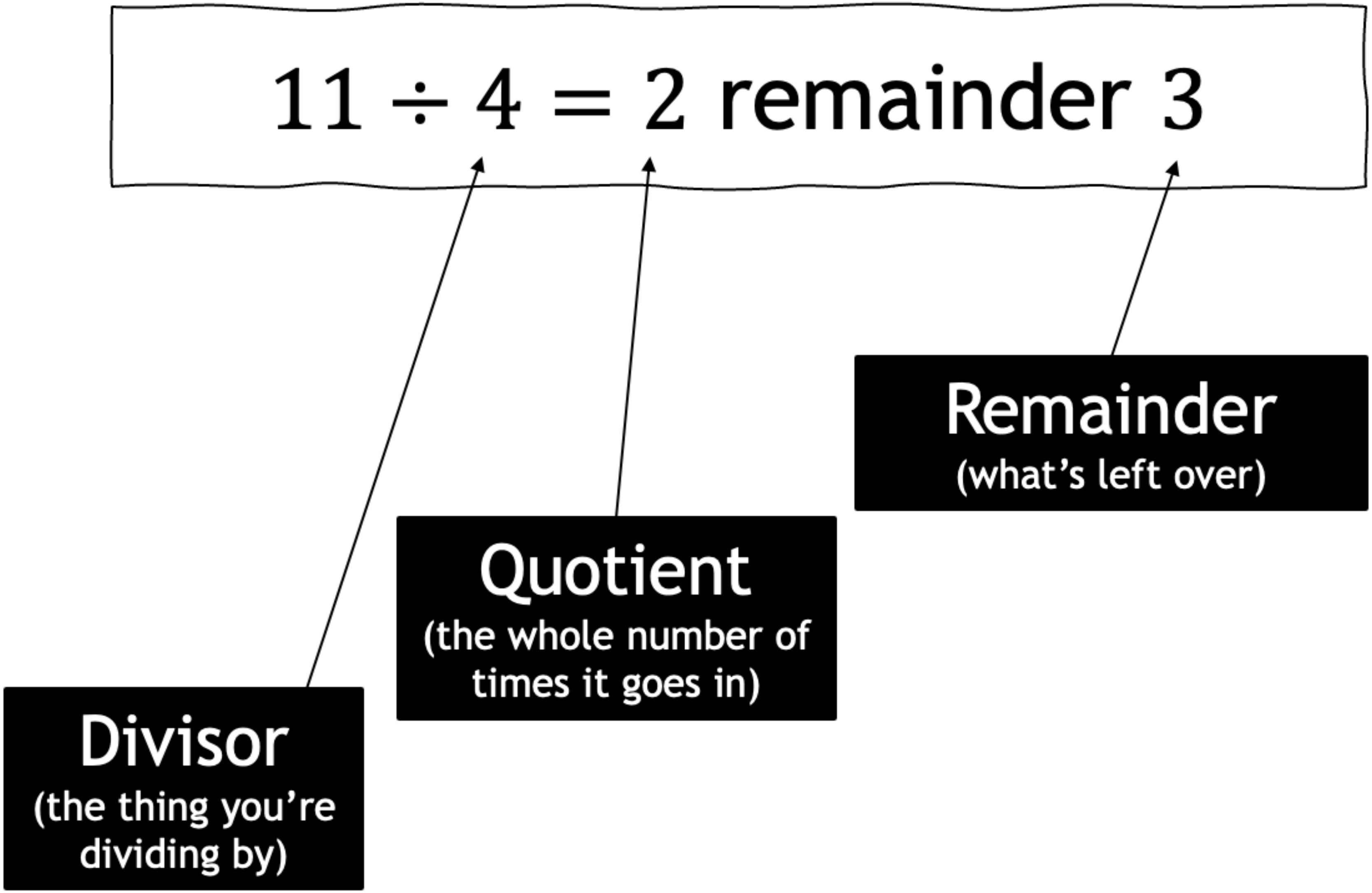
Worked Example	Your Turn
In the expansion of $(p + qx)^5$, where p is a positive constant and q is a negative constant, the numerical term is 3125 and the coefficient of x^2 is 31250. Hence find the values of p and q	In the expansion of $(py + q)^6$, where p and q are positive constants, the coefficient of y^2 is 34560 and the coefficient of y^6 is 729. Hence find the values of p and q.

Worked Example	Your Turn
In the expansion of $(p+qx)^6$, where p is a positive constant and q is a negative constant, the coefficient of x^2 is 4860 and the coefficient of x^3 is -4320. Hence find the values of p and q	In the expansion of $(p+qy)^7$, where p is a positive constant and q is a negative constant, the coefficient of y is -10206 and the coefficient of y^2 is 20412. Hence find the values of p and q

Extra Notes

2 Factor Theorem

Terminology



Terminology

$$\frac{x^2 - 4}{x - 2} = x + 2$$

The **quotient** is $x + 2$

The **remainder** is 0

The **divisor** is $x - 2$

Since $x^2 - 4$ divides exactly by $x - 2$, there is no remainder.

Why is Algebraic Division Useful?

In a moment we'll see how to **divide algebraic expressions**, where there might be a remainder, or simplifying by factorisation is difficult. But **why** might we want to do so?

$$11 \div 4 = 2 \text{ remainder } 3$$

$$\frac{11}{4} = 2\frac{3}{4}$$

For normal numbers, we can use to turn an **improper fraction** into a **mixed number**.

$$\frac{x+3}{x+1}$$

$$1 + \frac{2}{x+1}$$

We can do the same for algebraic division. We will see why this form might be more convenient, e.g. when trying to **integrate**.

Algebraic Division

Worked Example	Your Turn
Given that $2x^3 + 5x^2 - 11x + 4 = (2x - 1)(Ax^2 + Bx + C)$ Use algebraic long division to work out the values of the constants A, B, and C.	Given that $2x^3 + 3x^2 - 29x + 30 = (2x - 3)(Ax^2 + Bx + C)$ Use algebraic long division to work out the values of the constants A, B, and C.

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Worked Example	Your Turn
Given that $3x^4 + 11x^3 - 14x^2 - 32x + 32 = (3x - 4)(Ax^3 + Bx^2 + Cx + D)$ Work out the values of the constants A, B, C, and D.	Given that $2x^4 + x^3 - 16x^2 + 3x + 18 = (2x - 3)(Ax^3 + Bx^2 + Cx + D)$ Work out the values of the constants A, B, C, and D.

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Factor Theorem

Consider the following factorisation of a cubic expression:

Let $f(x) = x^3 + x^2 - 4x - 4$
$\equiv (x-2)(x^2+3x+2)$

What is $f(2)$?

$$f(2) = (2-2)(2^2 + 3(2) + 2)$$
$$= 0$$

From the above example, this statement is trivial. But <u>is the converse true</u>?
i.e. if $f(a) = 0$, then is $(x-a)$ a factor?
The answer is yes...

Complete the more general statement...

If $(x-a)$ is a factor of a polynomial, then $\boldsymbol{f(a) = 0}$

Factor Theorem

The factor theorem: if $(x - a)$ is a factor of a polynomial $f(x)$, then $f(a) = 0$

Worked Example	Your Turn
Given that $$f(x) = 4x^3 - 28x^2 + 35x + 25$$ Select which of the following are factors of $f(x)$ ○ $(2x+1)$ ○ $(2x-5)$ ○ both ○ neither	Given that $$f(x) = 4x^3 - 20x^2 + 13x + 12$$ Select which of the following are factors of $f(x)$ ○ $(2x+1)$ ○ $(2x-3)$ ○ both ○ neither

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Worked Example	Your Turn
Given that $(2x+1)$ is a factor, factorise $f(x) = 18x^3 - 51x^2 + 20x + 25$	Given that $(2x+3)$ is a factor, factorise $f(x) = 6x^3 + 23x^2 + 9x - 18$

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Fluency Practice

Cubic Expression	Is $(x-1)$ a factor?	Is $(x-2)$ a factor?	Is $(x+1)$ a factor?	Is $(x+3)$ a factor?	Factorised Expression
$x^3+2x^2-13x+10$	Yes	Yes	No	No	$(x-1)(x-2)(x+5)$
x^3-5x^2+2x+8	No	Yes	Yes	No	$(x-2)(x+1)$
$x^3-2x^2-21x-18$	No	No	Yes	Yes	$(x+3)(x+1)$
$x^3+x^2-14x+24$		Yes			$(x-2)$
$x^3-10x^2+23x-14$	Yes				$(x-1)$
x^3+8x^2-x-8					
$x^3-4x^2-11x+30$					
$x^3-x^2-16x+16$					
$x^3+3x^2-18x-40$					
$x^3-8x^2+13x-6$					
$2x^3+5x^2-23x+10$					

Worked Example	Your Turn
$f(x) = 4x^3 + bx^2 + 33x + 18$ where b is a constant Given that $(2x + 3)$ is a factor of $f(x)$, find the value of b.	$f(x) = 4x^3 + bx^2 - 15x - 4$ where b is a constant Given that $(2x + 1)$ is a factor of $f(x)$, find the value of b.

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Worked Example	Your Turn
Given that $(x+4)$ and $(x-1)$ are factors of $f(x) = 3x^3 + rx^2 + 9x + s$, determine the values of the constants r and s	Given that $(x+5)$ and $(x-1)$ are factors of $f(x) = 2x^3 + mx^2 + nx - 35$, determine the values of the constants m and n

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Worked Example	Your Turn
Solve $8x^3 - 14x^2 + 3x = 0$	Solve $3x^3 + 7x^2 - 6x = 0$

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Worked Example	Your Turn
Given that $x = -\frac{1}{2}$ is a solution to the equation $2x^3 - 9x^2 + 3x + 4 = 0$ Find all the solutions to the equation.	Given that $x = -\frac{3}{4}$ is a solution to the equation $8x^3 - 38x^2 + 27x + 45 = 0$ Find all the solutions to the equation.

Extra Notes