



KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS



KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

2025

Year 11

Level 2 Further Mathematics

2026

Unit 28 Tasks

DO NOT WRITE INSIDE

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1 Matrices

Write down the order of these matrices.

(a) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 3 \\ 2 & -2 \\ 4 & 1 \end{pmatrix}$

(c) $(3 \ 2 \ 6)$ (d) $\begin{pmatrix} 0.5 & 0 \\ 1.5 & 1 \end{pmatrix}$

Work out:

(a) $\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

(c) $\begin{pmatrix} -1 & 0 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 2 & -3 \\ 0 & -2 \end{pmatrix}$

(d) $\begin{pmatrix} 5 & 0.5 \\ -0.5 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ -4 & 0.5 \end{pmatrix}$

Work out:

(a) $2 \times \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ (b) $4 \times \begin{pmatrix} 3 & -1 \\ 0.5 & 6 \end{pmatrix}$

(c) $-3 \times \begin{pmatrix} 11 \\ -8 \end{pmatrix}$ (d) $\frac{1}{2} \times \begin{pmatrix} 4 & -6 \\ 2 & 0 \\ 3 & 8 \end{pmatrix}$

Given that

$$A = \begin{pmatrix} -2 & 3 \\ 0 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 4 & -1 \\ -3 & 7 \end{pmatrix} \quad C = \begin{pmatrix} -2 & 0 \\ 8 & -3 \end{pmatrix}$$

Find:

(a) $A + B - C$

(b) $C - B$

(c) $2A$

(d) $A + 2B$

(e) $3C + B$

(f) $4B - A$

(g) $2A + 3C$

(h) $-4B + A + 2B$

Is it possible to multiply the matrices shown?

(a) $\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \times \begin{pmatrix} 5 & -1 \\ 4 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} -7 & 4 \end{pmatrix} \times \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 \\ 4 & -3 \end{pmatrix} \times \begin{pmatrix} 3 & 2 & 5 \\ 6 & 0 & -1 \end{pmatrix}$

Work out:

(a) $\begin{pmatrix} 4 \\ 2 \end{pmatrix} \times \begin{pmatrix} -2 & 5 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 3 \\ 2 & 5 \end{pmatrix} \times \begin{pmatrix} -1 & 3 \\ 0 & 6 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 7 & -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & -2 \\ 3 & 7 \end{pmatrix} \times \begin{pmatrix} -1 & 4 \\ 0 & -2 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 2 \\ -5 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 6 \\ -3 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} -2 & 1 \\ 8 & 0 \end{pmatrix} \times \begin{pmatrix} -3 & 5 \\ 1 & 2 \end{pmatrix}$

(a) Given that

$$\begin{pmatrix} -2 & a \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 22 \\ 9 \end{pmatrix}$$

work out the value of a .

(b) Matrix $\mathbf{P} = \begin{pmatrix} 2 & 3 \\ a & b \end{pmatrix}$

Matrix $\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

You are given that $\mathbf{PQ} = \mathbf{QP}$. Work out the values of a and b .

Purposeful Practice

$$\textcircled{1} \mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 6 & -2 \\ -3 & -1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ -3 & -5 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Work out

- | | | |
|-----------|-----------|----------|
| (i) 4A | (ii) 2D | (iii) AF |
| (iv) CE | (v) DH | (vi) BH |
| (vii) AB | (viii) BA | (ix) BC |
| (x) CB | (xi) DA | (xii) BD |
| (xiii) AC | (xiv) DC | |

$\textcircled{2}$ Work out the value of p in each of the following.

$$(i) \begin{bmatrix} 4 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 2 & -1 \\ p & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$(iii) \begin{bmatrix} p & 1 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ p \end{bmatrix} = \begin{bmatrix} 2 \\ 17 \end{bmatrix}$$

$$(iv) \begin{bmatrix} p & 4p \\ p & -2p \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \end{bmatrix}$$

$$(v) \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ p & 4 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 16 & 9 \end{bmatrix}$$

$$(vi) \begin{bmatrix} 4 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2p \\ -1 & p \end{bmatrix} = \begin{bmatrix} 9 & -14 \\ 0 & 0 \end{bmatrix}$$

$\textcircled{3}$ Work out the values of x and y in each of the following.

$$(i) \begin{bmatrix} 2 & 1 \\ 1 & y \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 1 & x \\ 2y & 3y \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \end{bmatrix}$$

$$(iii) \begin{bmatrix} -3 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ x & y \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ -4 & 10 \end{bmatrix}$$

$$(iv) \begin{bmatrix} x & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ x & y \end{bmatrix} = \begin{bmatrix} -9 & -5 \\ -2 & -4 \end{bmatrix}$$

$$\textcircled{4} \text{ Given that } \begin{bmatrix} 5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

- (i) write down two equations in x and y
 (ii) work out x and y by solving the pair of simultaneous equations.

$\textcircled{5}$ Work out the values of a and b in each of the following.

$$(i) \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 21 \end{bmatrix}$$

$$(ii) \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = 6 \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$(iii) \begin{bmatrix} 3 & b \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & 2a \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 11 \\ 2 & -2 \end{bmatrix}$$

$$(iv) \begin{bmatrix} 5 & 1 \\ 3a & b+1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 12 & 3 \end{bmatrix}$$

PS $\textcircled{6}$ Given that $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, work out the values of a, b, c and d .

PS $\textcircled{7}$ $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 0 & k \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$. Work out the value of k such that $\mathbf{AB} = \mathbf{BA}$.

PS $\textcircled{8}$ Given that $\begin{bmatrix} p & 4 \\ 1 & q \end{bmatrix} \begin{bmatrix} r & p-4 \\ q+8 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 3p+4 \\ -17 & -3p \end{bmatrix}$, work out the values of p, q and r .

PS $\textcircled{9}$ (i) Calculate the product $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(ii) Hence write down the matrix \mathbf{M} , in terms of a, b, c and d , such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Discussion point

Are there any values of a, b, c and d , for which the matrix \mathbf{M} in question 9 does not exist?

$$A = \begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix} \quad B = \begin{pmatrix} -2 & 0 \\ 4 & 5 \end{pmatrix}$$

- (a) Given that $B + C = I$, find C
- (b) Given that $D - A = I$, find D
- (c) Given that $B + 2I = E$, find E

- (a) Given that

$$\begin{pmatrix} x & -2 \\ -7 & y \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} = I$$

Find the values of x and y .

- (b) Given that

$$\begin{pmatrix} 4 & -1 \\ -7 & 2 \end{pmatrix} \begin{pmatrix} 2 & p \\ q & 4 \end{pmatrix} = I$$

Find the values of p and q .

- (a) Find I^2

(b) Given that $2A + I^2 = \begin{pmatrix} 6 & -4 \\ -1 & 5 \end{pmatrix}$

find A .

(a) Given that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix} = I$

find the values of a, b, c and d .

(b) Given that $\begin{pmatrix} -5 & 3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = I$

find the values of a, b, c and d .

- (c) Given that

$$\begin{pmatrix} x & \frac{1}{2} \\ -2 & y \end{pmatrix} \begin{pmatrix} 1 & z \\ 2 & 4 \\ -\frac{3}{3} & -\frac{9}{9} \end{pmatrix} = I^2$$

find the values of x, y , and z .

Fluency Practice

1 Work out

(a) $\begin{pmatrix} 4 & 2 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 7 \\ 1 \end{pmatrix}$



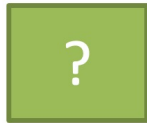
(b) $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} -3 \\ -4 \end{pmatrix}$



(c) $2 \begin{pmatrix} 5 & -2 \\ 6 & -3 \end{pmatrix}$



(d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix}$



(e) $6 \begin{pmatrix} -4 & 7 \\ -1 & -3 \end{pmatrix}$



(f) $\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ 6 \end{pmatrix}$



2

$A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$

$B = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$

$C = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$

Work out

(a) AB



(b) BC



(c) 3A



(d) BA



(e) -C



(f) $B \begin{pmatrix} 1 & -4 \\ -5 & 7 \end{pmatrix}$



3

$P = \begin{pmatrix} -2 & 0 \\ 5 & 1 \end{pmatrix}$

$Q = \begin{pmatrix} -4 & 1 \\ 3 & -2 \end{pmatrix}$

$C = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Work out

(a) P^2



(b) QP



(c) 5Q



(d) PC



(e) IQ



(f) 3I



Fluency Practice

4

<p>(a) $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 1 & -4 \end{pmatrix}$?</p>	<p>(b) $\begin{pmatrix} -3 & -2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 3 & 4 \end{pmatrix}$?</p>	<p>(c) $\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$?</p>
<p>(d) $\begin{pmatrix} 10 & -7 \\ 9 & 8 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ -2 & 3 \end{pmatrix}$?</p>	<p>(e) $\begin{pmatrix} 1 & -2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$?</p>	<p>(f) $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & -5 \end{pmatrix}$?</p>

5 Work out, giving your answers as simply as possible.

<p>(a) $\begin{pmatrix} \sqrt{2} & 1 \\ -1 & 3\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ -3 & -2\sqrt{2} \end{pmatrix}$?</p>	<p>(b) $\begin{pmatrix} -\frac{1}{2} & -1 \\ \frac{3}{2} & 5 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ -\frac{1}{2} & 3 \end{pmatrix}$?</p>	<p>(c) $\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}^2$?</p>
<p>(d) $\begin{pmatrix} 3\sqrt{3} & -4 \\ 2 & 3\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 1 \\ -4 & 0 \end{pmatrix}$?</p>	<p>(e) $\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$?</p>	<p>(f) $\begin{pmatrix} \sqrt{2} & 2 \\ 7 & \sqrt{3} \end{pmatrix}^2$?</p>

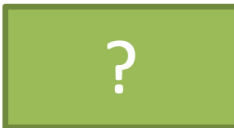
6 Work out, giving your answers as simply as possible.


<p>(a) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p \\ p+1 \end{pmatrix}$?</p>	<p>(b) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$?</p>	<p>(c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m \\ 2m \end{pmatrix}$?</p>
<p>(d) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}$?</p>	<p>(e) $\begin{pmatrix} 4t & 0 \\ 0 & 4t \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$?</p>	<p>(f) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix}$?</p>


Fluency Practice


7


Work out, giving your answers as simply as possible.


(a) $\begin{pmatrix} 2x & -3 \\ -5 & 4x \end{pmatrix} \begin{pmatrix} x & 3x \\ -3 & 0 \end{pmatrix}$ 

(b) $\begin{pmatrix} a & 3a \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 7 & 8 \\ -10 & 11 \end{pmatrix}$ 

(c) $\begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix}^2$ 

(d) $\begin{pmatrix} y & y \\ -3 & x \end{pmatrix} \begin{pmatrix} 2 & 3y \\ 0 & 1 \end{pmatrix}$ 

(e) $\begin{pmatrix} a+1 & a \\ a+2 & a+1 \end{pmatrix} \begin{pmatrix} a+1 & -a \\ -a-2 & a+1 \end{pmatrix}$ 

(f) $\begin{pmatrix} 3x & -3 \\ -9 & x+1 \end{pmatrix}^2$ 

Fluency Practice

1
$$\begin{pmatrix} -2 & a \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 22 \\ 9 \end{pmatrix}$$



Work out the value of a .

2 June 2013 Paper 2 Q11

(a) Work out
$$\begin{pmatrix} 2 & -1 \\ \frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ a & c \end{pmatrix}$$



Give your answer in terms of a , b and c .

(b) You are given that
$$\begin{pmatrix} 2 & -1 \\ \frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ a & c \end{pmatrix} = I$$
 where I is the identity matrix.

Work out the values of a , b and c .

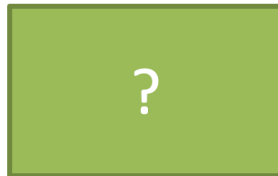


3 Set 2 Paper 2 Q16

Matrix $P = \begin{pmatrix} 2 & 3 \\ a & b \end{pmatrix}$ Matrix $Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

You are given that $PQ = QP$

Work out the values of a and b .



4 Set 4 Paper 1 Q17

$$\begin{pmatrix} 2 & a \\ 1 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Work out **all** possible pairs of values of a and b .



Problem Solving

$$PQ = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

$$QP = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix}$$

2x2 matrix multiplication is *not* necessarily commutative

BUT, can you find 2x2 matrices A and B such that $AB = BA$? (it is possible!)

Purposeful Practice

- ① Work out the image of point (4, 2) for the transformation defined by matrix $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.
- ② Work out the image of point (1, -3) for the transformation represented by matrix $\begin{bmatrix} 0 & -3 \\ -1 & 5 \end{bmatrix}$.
- ③ Work out the image of point (-2, -3) for the transformation defined by matrix $\begin{bmatrix} -2 & -3 \\ 2 & -1 \end{bmatrix}$.

④ The image of point (4, 3) under the transformation matrix $\begin{bmatrix} 2 & 1 \\ c & 3 \end{bmatrix}$ is (11, 1). Work out the value of c .

⑤ The image of point (a, 1) under the transformation matrix $\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$ is (7, 17). Work out the value of a .

⑥ The image of point (3, -2) under the transformation matrix $\begin{bmatrix} a & 2a \\ b & 3 \end{bmatrix}$ is (b, b). Work out the values of a and b .

⑦ The transformation matrix $\begin{bmatrix} 2c & d \\ c & -d \end{bmatrix}$ maps the point (2, 5) to the point (-6, 12). Work out the values of c and d .

⑧ Given that $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, show that $\mathbf{A}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

\mathbf{A}^2 means $\mathbf{A} \times \mathbf{A}$, i.e. \mathbf{AA}

⑨ Given that $\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 7 & p \end{bmatrix} = \mathbf{I}$, work out the value of p .

PS ⑩ Under the transformation matrix $\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, point D is mapped to (11, 10). Work out the coordinates of D.

PS ⑪ The matrices $\mathbf{M} = \begin{bmatrix} a & 3 \\ 2 & b \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} a+1 & b+2 \\ -a & 2 \end{bmatrix}$ satisfy the equation $\mathbf{MN} = \mathbf{I}$. Work out the values of a , b and c .

PS

12

(i) Under the transformation matrix $\begin{bmatrix} 4 & -1 \\ 6 & -1 \end{bmatrix}$, which of the following points is **not** invariant?


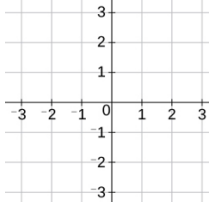
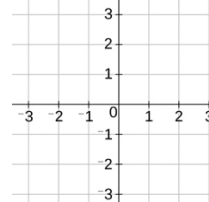
(1, 3) (2, 4) (3, 9) (5, 15)

(ii) If the point (x, y) is invariant under $\begin{bmatrix} 4 & -1 \\ 6 & -1 \end{bmatrix}$, identify a condition for all of the invariant points of this transformation.

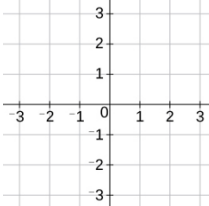
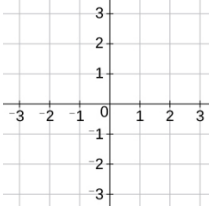
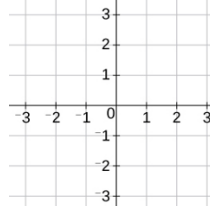
Note

Question 12 is for interest only as this specification does not assess a candidate's knowledge of invariant points, i.e. a point which does not move (see GCSE Maths).

Purposeful Practice

Rotations Using Matrices		
<p>(a)</p> <p>By considering the unit square, determine the matrix which describes a rotation 90° clockwise about the origin.</p>	<p>(b)</p> <p>Describe fully the single transformation represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$</p>	<p>(c)</p> <p>By considering the unit square, determine the matrix which describes a rotation 180° about the origin.</p>
		
<p>(d)</p> <p>The point $(1, -6)$ is mapped onto the point (a, b) when rotated 90° anti-clockwise about the origin. Using matrix algebra, find the values of a and b.</p>	<p>(e)</p> <p>The point (c, d) is mapped onto the point $(2, 4)$ when rotated 270° anti-clockwise about the origin. Using matrix algebra, find the values of c and d.</p>	<p>(f)</p> <p>A triangle with vertices at $(1, 1)$, $(5, 2)$ and $(4, -1)$ is rotated 180° about the origin. Use matrix algebra to find the coordinates of the vertices of the rotated triangle.</p>
<p>(g)</p> <p>Use matrix algebra to show that a rotation of 90° clockwise about the origin, followed by a rotation of 180° is equivalent to a rotation of 90° anti-clockwise about the origin.</p>	<p>(h)</p> <p>The point $(a, 6)$ is mapped onto the point $(b, -4)$ following a rotation of 90° anti-clockwise about the origin. Use matrix algebra to find the values of a and b.</p>	<p>(i)</p> <p>The point $(x, 2y + 6)$ is mapped onto the point $(2x, y - 7)$ following a rotation of 90° clockwise about $(0, 0)$. Use matrix algebra to find the values of x and y.</p>

Purposeful Practice

Reflections Using Matrices		
<p>(a)</p> <p>By considering the unit square, determine the matrix which describes a reflection in the x-axis.</p>	<p>(b)</p> <p>Describe fully the single transformation represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$</p>	<p>(c)</p> <p>By considering the unit square, determine the matrix which describes a reflection in the line $y = -x$.</p>
		
<p>(d)</p> <p>The point $(-4, 2)$ is mapped onto the point (a, b) when reflected in the x-axis. Using matrix algebra, find the coordinates (a, b).</p>	<p>(e)</p> <p>The point (c, d) is mapped onto the point $(7, -5)$ when reflected in the line $y = -x$. Using matrix algebra, find the coordinates (c, d).</p>	<p>(f)</p> <p>A triangle with vertices at $(0, 5)$, $(4, 3)$ and $(1, -1)$ is reflected in the line $y = x$. Use matrix algebra to find the coordinates of the vertices of the reflected triangle.</p>
<p>(g)</p> <p>A triangle with vertices at $(0, 1)$, $(1, 0)$ and $(3, 2)$ is reflected so its vertices map to $(0, -1)$, $(-1, 0)$ and $(-2, -3)$. Find the transformation matrix and the line of reflection.</p>	<p>(h)</p> <p>The point $(-2, a)$ is mapped onto the point $(b, 3)$ following a reflection in the line $x = 0$. Use matrix algebra to find the values of a and b.</p>	<p>(i)</p> <p>The point $(x, 3x - 7)$ is mapped onto the point $(y + 3, y)$ following a reflection in the line y-axis. Use matrix algebra to find the values of x and y.</p>

A triangle with coordinates $(3,2)$, $(5,2)$ and $(3,6)$ is transformed by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. By pre-multiplying, find the coordinates of the transformed triangle. Draw this transformation on a grid and hence describe it fully.

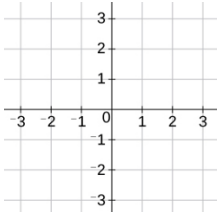
A triangle with coordinates $(-3,2)$, $(-5,2)$ and $(-3,5)$ is transformed by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. By pre-multiplying, find the coordinates of the transformed triangle. Draw this transformation on a grid and hence describe it fully.

A triangle with coordinates $(2,3)$, $(4,3)$ and $(4,7)$ is transformed by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. By pre-multiplying, find the coordinates of the transformed triangle. Draw this transformation on a grid and hence describe it fully.

A triangle with coordinates $(3,1)$, $(5,1)$ and $(3,5)$ is transformed by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. By pre-multiplying, find the coordinates of the transformed triangle. Draw this transformation on a grid and hence describe it fully.

The transformation matrix $\begin{pmatrix} a & 2 \\ -1 & 1 \end{pmatrix}$ maps the point $(3, 4)$ onto the point $(2, b)$. Work out the values of a and b .

Purposeful Practice

Enlargements Using Matrices		
(a) By considering the unit square, determine the matrix which describes an enlargement about the origin with scale factor 3.	(b) Describe fully the single transformation represented by the matrix $\begin{pmatrix} \frac{5}{2} & 0 \\ 0 & \frac{5}{2} \end{pmatrix}$	(c) Use matrix algebra to show that an enlargement of scale factor 2 about $(0, 0)$, followed by an enlargement of scale factor 1.5 about $(0, 0)$ is equivalent to an enlargement of scale factor 3 about $(0, 0)$.
		
(d) The point $(-5, 3)$ is mapped onto the point (a, b) when enlarged by a scale factor 2 about the origin. Using matrix algebra, find the values of a and b .	(e) The unit square OABC with coordinates $O(0, 0)$, $A(0, 1)$, $B(1, 1)$ and $C(1, 0)$ is mapped to $OA'B'C'$ under matrix $\begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}$. Use matrix algebra to find the coordinates of A' , B' and C' .	(f) The point (c, d) is mapped onto the point $(-1, -4)$ when enlarged by a scale factor 0.5 about the origin. Using matrix algebra, find the values of c and d .
(g) Use matrix algebra to show that an enlargement of scale factor 2 about $(0, 0)$, followed by an enlargement of scale factor -0.5 about $(0, 0)$ is the same as a rotation of 180° about the origin.	(h) The point $(a, 3)$ is mapped to the point $(6, 2a)$ when enlarged with scale factor b about the origin. Use matrix algebra to find the possible values of a and b .	(i) The point $(x - 4, y)$ is mapped to the point $(2y, 2x - 18.5)$ when transformed under the matrix $\begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}$. Find the values of x and y .

Purposeful Practice

① Work out the 2×2 matrix that represents each of the following transformations.

- (i) Reflection in the x -axis.
- (ii) Rotation of 90° about O .
- (iii) Enlargement, scale factor 2, centre the origin.
- (iv) Reflection in the y -axis.
- (v) Reflection in the line $y = x$.
- (vi) Rotation by 180° , centre the origin.
- (vii) Reflection in the line $y = -x$.
- (viii) Enlargement, scale factor -3 , centre O .
- (ix) Enlargement, centre O , scale factor $\frac{1}{2}$.

② The unit square $OABC$ is transformed by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to $OA'B'C'$. Show the image on a diagram, labelling each vertex.

③ The unit square $OABC$ is transformed by the matrix $\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ to $OA'B'C'$. Show the image on a diagram, labelling each vertex.

④ Describe fully the transformations given by the following matrices.

(i) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(vii) $\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$ (viii) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

⑤ The unit square $OABC$ is transformed to $OA'B'C'$. $OA'B'C'$ is shown on the diagram.

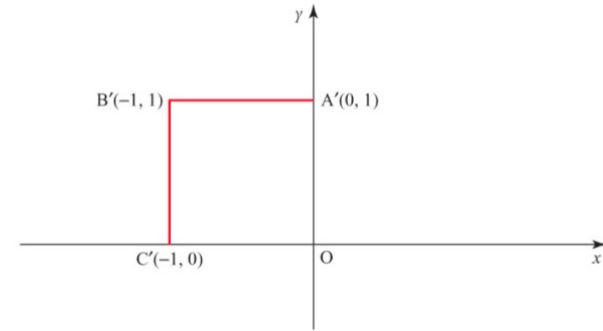


Figure 9.5

Work out the matrix for the transformation.

⑥ The unit square $OABC$ is transformed by the matrix $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ to $OA'B'C'$.

Work out the area of $OA'B'C'$.

⑦ The unit square $OABC$ is transformed by the matrix $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ to $OA'B'C'$. The area of $OA'B'C'$ is 64 square units.

Work out the two possible values of k .

⑧ (i) Draw a diagram to show the unit square $OABC$ rotated 45° about the origin.

(ii) Work out the coordinates of A' and C' (the images of A and C).

(Hint: $\sin 45^\circ = \frac{\sqrt{2}}{2}$ and $\cos 45^\circ = \frac{\sqrt{2}}{2}$.)

(iii) Hence write down the transformation matrix for a rotation of 45° about the origin.

Note

Question 8 is for interest only as this specification only includes rotations of 90° , 180° and 270° .

Purposeful Practice

- 1 [Jan 2013 Paper 2 Q15] Describe fully the **single** transformation represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

?

- 2 [Set 2 Paper 1 Q4] The transformation matrix $\begin{pmatrix} a & 2 \\ -1 & 1 \end{pmatrix}$ maps the point (3,4) onto the point (2, b). Work out the values of a and b .

?

- 3 [Set 3 Paper 1 Q6] The matrix $\begin{pmatrix} a & b \\ -a & 2b \end{pmatrix}$ maps the point (5,4) onto the point (1,17). Work out the values of a and b .

?

- 4 [Worksheet 2 Q5] Work out the image of the point $D(-1, 2)$ after transformation by the matrix $\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$

?

- 5 [Worksheet 2 Q6] The point $A(m, n)$ is transformed to the point $A'(-2, 0)$ by the matrix $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$
Work out the values of m and n .

?

- 6 [Worksheet 2 Q8] Describe fully the transformation given by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

?

- 7 [Worksheet 2 Q9] The unit square $OABC$ is transformed by the matrix $\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$ to the square $OA'B'C'$. The area of $OA'B'C'$ is 27. Work out the exact value of h .

?

Purposeful Practice

Harder Transformations Using Matrices		
(a)	(b)	(c)
Find the single matrix that represents an enlargement about the origin with scale factor 3, followed by a rotation of 90° clockwise about the origin.	Find the single matrix that represents a reflection in the y -axis, followed by a rotation of 180° about the origin.	$P = \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$ <p>Matrices P and Q represent different transformations. Find the single matrix that represents transformation P followed by transformation Q.</p>
(d)	(e)	(f)
The point P $(4, -2)$ is mapped to the point Q following a reflection in the line $y = x$, then an enlargement with scale factor 2 about the origin. Use matrix algebra to find the coordinates of point Q.	The point (a, b) is mapped to the point $(-5, 1)$ following a rotation of 180° about the origin, then a reflection in the x -axis. Using matrix algebra, find the coordinates (a, b) .	The matrix $\begin{pmatrix} 0 & b \\ -2 & 4 \end{pmatrix}$ maps the point $(a, -3)$ onto the point $(-9, 5)$. Use matrix algebra to find the values of a and b .
(g)	(h)	(i)
The transformation matrix $\begin{pmatrix} a & 2b \\ -a & 3 \end{pmatrix}$ maps the point $(2, -1)$ to the point $(6, 7)$. Find the values of a and b .	The transformation matrix $\begin{pmatrix} b & 2a \\ a & -b \end{pmatrix}$ maps the point $(6, 3)$ to the point $(24, b)$. Find the values of a and b .	Point $(c, 4)$ is mapped to the point $(-2, d)$ by the transformation matrix $\begin{pmatrix} c & -3 \\ 2 & -1 \end{pmatrix}$. Use matrix algebra to find the two possible values of c and d .

Find the matrices that represent the following transformations:

- (a) A reflection in the x -axis, followed by a rotation through 180° centre the origin.
- (b) An enlargement with centre the origin and scale factor 2, followed by a reflection in the line $y = x$.
- (c) A reflection in the y -axis followed by a reflection in the line $y = x$.
- (d) A reflection in the line $y = x$ followed by enlargement about the origin with scale factor 3.

Point $(3, -2)$ is transformed by the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ followed by a further transformation by the matrix $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

- (i) Work out the matrix for the combined transformation.
- (ii) Work out the co-ordinates of the image point of P .

Point $(-1, 4)$ is transformed by the matrix $\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}$ followed by a further transformation by the matrix $\begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$.

- (i) Work out the matrix for the combined transformation.
- (ii) Work out the co-ordinates of the image point of W .

The transformation matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ maps a point P to Q . The transformation matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ maps point Q to point R . Point R is $(-4, 3)$. Work out the coordinates of point P .

Purposeful Practice

- ① Point $P(3, -2)$ is transformed by $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, followed by a further

transformation $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$.

- (i) Work out the matrix for the combined transformation.
 (ii) Work out the x -coordinate of the image point of P .

- ② Point $W(-1, 4)$ is transformed by $\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$, followed by a further

transformation $\begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$.

- (i) Work out the matrix for the combined transformation.
 (ii) Work out the coordinates of the image point of W .

- ③ The unit square is reflected in the x -axis followed by a rotation through 180° , centre the origin.

Work out the matrix for the combined transformation.

- ④ The unit square is enlarged, centre the origin, scale factor 2, followed by a reflection in the line $y = x$.

Work out the matrix for the combined transformation.

- ⑤ The unit square is rotated by 90° , centre the origin, followed by a reflection in the y -axis.

Work out the matrix for the combined transformation.

⑥ Matrix $\mathbf{P} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

- (i) Describe the transformation that this matrix represents.
 (ii) Work out \mathbf{P}^2 .
 (iii) Use transformations to interpret your answer to part (iii).

⑦ Matrix $\mathbf{D} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Matrix $\mathbf{E} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

- (i) Describe the transformation that matrix \mathbf{D} represents.
 (ii) Describe the transformation that matrix \mathbf{E} represents.
 (iii) Work out \mathbf{DE} .
 (iv) Describe the transformation that matrix \mathbf{DE} represents.
 (v) Use transformations to interpret your answer to part (iv).
 (vi) Use transformations to explain why $\mathbf{ED} = \mathbf{DE}$.

- PS** ⑧ Use matrices to prove the following:

enlargement, centre O , scale factor 3 followed by enlargement, centre O , scale factor -2 is equivalent to a single enlargement, centre O , scale factor k , where k is an integer to be found.

- PS** ⑨ (i) Calculate the combined transformation matrix of $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$, followed by $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, and finally $\begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$.

- (ii) Under this combined transformation, point A is mapped to the point $(-14, 7)$. Work out the coordinates of point A .

- PS** ⑩ Identify three matrices \mathbf{A} , \mathbf{B} and \mathbf{C} such that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

Purposeful Practice

1 Point $(3, -2)$ is transformed by the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ followed by a further transformation by the matrix $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

(i) Work out the matrix for the combined transformation.

(ii) Work out the co-ordinates of the image point of P .

2 Point $(-1, 4)$ is transformed by the matrix $\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}$ followed by a further transformation by the matrix $\begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$.

(i) Work out the matrix for the combined transformation.

(ii) Work out the co-ordinates of the image point of W .

3 The unit square is reflected in the x -axis followed by a rotation through 180° centre the origin. Work out the matrix for the combined transformation.

4 The unit square is enlarged, centre the origin, scale factor 2 followed by a reflection in the line $y = x$. Work out the matrix for the combined transformation.

Purposeful Practice

- 5 [Jan 2013 Paper 2 Q17] $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ represents a reflection in the y -axis. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents a reflection in the line $y = x$.
Work out the matrix that represents a reflection in the y -axis followed by a reflection in the line $y = x$.

?

- 6 [June 2012 Paper Q22] The transformation matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ maps a point P to Q . The transformation matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ maps point Q to point R .
Point R is $(-4,3)$. Work out the coordinates of point P .

?

- 7 [Set 1 Paper Q14b] The unit square $OABC$ is transformed by reflection in the line $y = x$ followed by enlargement about the origin with scale factor 2. What is the matrix of the combined transformation?

?

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The point $P(2,7)$ is transformed by matrix BA to P' . Show that P' lies on the line $7x + 2y = 0$.

?

2 Trigonometric Identities and Equations

(a) Show that

$$\cos^2 \theta - \sin^2 \theta \equiv 1 - 2 \sin^2 \theta$$

(b) Show that $\tan^2 \theta \equiv \frac{\sin^2 \theta}{1 - \sin^2 \theta}$

(a) Prove that

$$\frac{1}{\cos x} - \cos x \equiv \sin x \tan x$$

(b) Show that $1 + \tan^2 \theta \equiv \frac{1}{\cos^2 \theta}$

(a) Prove that

$$\frac{\sin^4 x + \sin^2 x \cos^2 x}{\cos^2 x - 1} \equiv -1$$

(b) Prove that

$$(2 \sin x - \cos x)^2 + (\sin x + 2 \cos x)^2 \equiv 5$$

(a) Show that

$$\tan x + \frac{1}{\tan x} \equiv \frac{1}{\sin x \cos x}$$

(b) Show that

$$\frac{1}{\cos^2 x} - 1 \equiv \tan^2 x$$

THE GREEK PHILOSOPHER,

1	2	3	4	5	6	5	7	8
---	---	---	---	---	---	---	---	---

SAID THAT EVERYTHING WAS MADE UP OF THE FOUR SUBSTANCES,

			8	1	2	5	9	1	3	2	
10	3	2	8	1	11	12	13	1	5	8	2

Simplify the expressions below and find the answers in the code table. The letter with each answer and the question number give the code to find the puzzle solution.

1. $\frac{\sin^2\theta - \cos^2\theta}{\sin\theta + \cos\theta}$ =	8. $\frac{\sin^2\theta + \sin^2\theta \cos\theta + \cos^2\theta + \cos^3\theta}{\cos^2\theta + \cos^3\theta}$ =	F $\sin^2\theta$
2. $\cos^4\theta - \sin^4\theta$ =	9. $\frac{\sin\theta \cos^2\theta}{\cos^3\theta}$ =	I $\sin\theta$
3. $\sin^3\theta + \sin\theta \cos^2\theta$ =	10. $(\sin^3\theta + \sin\theta \cos^2\theta)^2$ =	E $1 + \cos\theta$
4. $\tan^2\theta + 1$ =	11. $\frac{\sin^2\theta - \cos^2\theta}{\sin\theta - \cos\theta}$ =	L $\cos\theta$
5. $\sin\theta \tan\theta + \cos\theta$ =	12. $\frac{\sin^2\theta}{\cos^2\theta}$ =	A $\sin\theta - \cos\theta$
6. $\frac{\tan\theta \cos\theta}{\sin\theta}$ =	13. $2 \cos^2\theta + 2 \sin^2\theta$ =	H $\tan\theta$
7. $\frac{\cos^2\theta \tan\theta \sin\theta + \cos^3\theta}{\cos^3\theta}$ =		O 1
		N $\cos\theta + \sin\theta$
		R $\cos^2\theta - \sin^2\theta$
		S $\frac{1}{\cos^2\theta}$
		T $\frac{1}{\cos\theta}$
		D $\tan^2\theta$
		W 2

A MEMBER OF THE ANIMAL KINGDOM THAT HAS AN INTERNAL SKELETON.

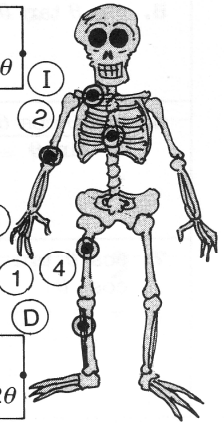
1	2	3	4	5
---	---	---	---	---

Simplify the trigonometric expressions. Each answer and the letter next to each question gives the code to find the human bone names. Join the dot next to each bone name to the dot on the bone on the skeleton. Each line will pass through a letter and number giving the puzzle answer code.

M	$2 \sin \theta + 3 \sin \theta =$	$5 \tan \theta - 2 \tan \theta =$	C
F	$2 \sin \theta \times 3 \sin \theta =$	$5 \cos^2 \theta \times \cos \theta =$	S
O	$\cos^2 \theta + 3 \cos^2 \theta =$	$\cos \theta + 5 \cos \theta =$	I
H	$10 \tan^5 \theta \div 2 \tan^4 \theta =$	$5 \tan 2 \theta - 4 \tan 2 \theta =$	R
N	$5 \tan \theta + 3 \tan \theta =$	$6 \sin^2 \theta - \sin^2 \theta =$	V
U	$4 \tan \theta \times 2 \tan \theta =$	$12 \cos 2 \theta - 8 \cos 2 \theta =$	T
L	$4 \sin^2 \theta - \sin^2 \theta =$	$\sin 2 \theta + 2 \sin 2 \theta =$	E
B	$3 \cos 2 \theta \times \cos 2 \theta =$	$12 \sin^3 \theta \div 3 \sin \theta =$	A

HUMAN BONES:

$5 \tan \theta$	$8 \tan^2 \theta$	$5 \sin \theta$	$3 \sin 2 \theta$	$\tan 2 \theta$	$4 \cos^2 \theta$	$8 \tan^2 \theta$	$5 \cos^3 \theta$
$6 \sin^2 \theta$	$6 \cos \theta$	$3 \cos^2 2 \theta$	$8 \tan^2 \theta$	$3 \sin^2 \theta$	$4 \sin^2 \theta$		
$6 \sin^2 \theta$	$3 \sin 2 \theta$	$5 \sin \theta$	$8 \tan^2 \theta$	$\tan 2 \theta$			
$5 \cos^3 \theta$	$4 \cos 2 \theta$	$3 \sin 2 \theta$	$\tan 2 \theta$	$8 \tan \theta$	$8 \tan^2 \theta$	$5 \sin \theta$	
$3 \tan \theta$	$3 \sin^2 \theta$	$4 \sin^2 \theta$	$5 \sin^2 \theta$	$6 \cos \theta$	$3 \tan \theta$	$3 \sin^2 \theta$	$3 \sin 2 \theta$



Fluency Practice

1 Solve the following in the range

$$0^\circ \leq x < 360^\circ$$

a $\sin \theta = 3 \cos \theta \rightarrow \theta =$

b $2 \sin \theta = 3 \cos \theta \rightarrow \theta =$

2 Solve the following by first factorising, $0^\circ \leq x < 360^\circ$

a $\cos^2 \theta - \cos \theta = 0 \rightarrow \theta =$

b $\tan^2 \theta - 3 \tan \theta = 0 \rightarrow \theta =$

c $\sin x \cos x + \sin x = 0 \rightarrow \theta =$

3 Solve the following: $0^\circ \leq x < 360^\circ$

a $\sin^2 \theta = \frac{3}{4} \rightarrow \theta =$

b $\cos^2 \theta = \frac{3}{4} \rightarrow \theta =$

c $\tan^2 \theta = 3 \rightarrow \theta =$

4 By factorising these 'quadratics', solve in the range $0 < x < 360$

a $3 \cos^2 \theta + 2 \cos \theta - 1 = 0 \rightarrow \theta =$

b $6 \sin^2 \theta - \sin \theta - 1 = 0 \rightarrow \theta =$

 $\sin \theta \cos \theta + \sin \theta + \cos \theta = -1 \rightarrow$

Fluency Practice

1 Simplify $3 \sin x (\sin x + 2) - 3(2 \sin x - \cos^2 x)$

$$=$$
$$=$$

?

2 Write out the following in terms of $\sin x$:

a) $\cos^2 x \tan^2 x =$

?

b) $\tan x \cos^3 x =$

?

c) $\cos x (2 \cos x - 3 \tan x) =$

?

3 Prove the following:

a. $\tan x \sqrt{1 - \sin^2 x} \equiv \sin x$

b. $\frac{1 - \cos^2 x}{1 - \sin^2 x} \equiv \tan^2 x$

c. $(1 + \sin x)(1 - \sin x) \equiv \cos^2 x$

d. $\frac{2 \sin x \cos x}{\tan x} \equiv 2 - 2 \sin^2 x$

e. $\frac{1}{\cos \theta} - \cos \theta \equiv \sin \theta \tan \theta$

f. $(2 \sin \theta - \cos \theta)^2 + (\sin \theta + 2 \cos \theta)^2 \equiv 5$

Solve for $0 \leq x < 360^\circ$, giving your answers to 1 decimal place.

- (a) $\sin x = 0.78$
- (b) $2 \tan x = 5$
- (c) $16 \cos^2 x = 9$
- (d) $\tan x + 2 = 5$
- (e) $4 \sin^2 x - 1 = 0$

Solve for $0 \leq x < 360^\circ$, giving your answers to 1 decimal place.

- (a) $5 \sin x = \cos x$
- (b) $\sin x + \cos x = 0$
- (c) $7 \sin x = 3 \tan x$

Solve for $0 \leq \theta < 360^\circ$, giving your answers to 1 decimal place.

- (a) $\sin^2 \theta - 3 \cos^2 \theta = 0$
- (b) $2 \cos^2 \theta - \cos \theta - 1 = 0$
- (c) $\tan^2 \theta + 3 \tan \theta = 0$
- (d) $2 \sin^2 \theta + 7 \sin \theta + 5 = 0$

Solve for $0 \leq \theta < 360^\circ$, giving your answers to 1 decimal place.

- (a) $2 \cos^2 \theta + 3 \sin \theta = 3$
- (b) $2 \sin^2 \theta = 3 - 3 \cos \theta$

Fluency Practice

Patrons are reminded that drawing the relevant graph is a good idea and that $\tan \theta \equiv \frac{\sin \theta}{\cos \theta}$. Also, $(\cos \text{ something})^2$ is written as $\cos^2 \text{ something}$.

Give all answers as either exact or to 1 decimal place.

1. Solve $\sin \theta = \frac{1}{2}$ for $0^\circ < \theta < 720^\circ$.
2. Solve $\cos \theta = \frac{\sqrt{3}}{2}$ for $-360^\circ < \theta < 360^\circ$.
3. Solve $\sin \theta = \frac{1}{\sqrt{2}}$ for $-360^\circ < \theta < 360^\circ$.
4. Solve $\tan \theta = \frac{1}{\sqrt{3}}$ for $0^\circ < \theta < 360^\circ$.
5. Solve $2 \sin \theta + 1 = 0$ for $0^\circ < \theta < 720^\circ$.
6. Solve $\cos^2 \theta = 1$ for $0^\circ < \theta < 720^\circ$.
7. Solve $\sin^2 \theta - 3 = 1$ for $-360^\circ < \theta < 360^\circ$.
8. Solve $\sin \theta = \frac{2}{3}$ for $0^\circ < \theta < 360^\circ$.
9. Solve $2 \tan \theta + 1 = 6$ for $-360^\circ < \theta < 360^\circ$.
10. Solve $3 \sin \theta + 1 = 0$ for $0^\circ < \theta < 360^\circ$.
11. Solve $3 \sin \theta = 5 \cos \theta$ for $0^\circ < \theta < 720^\circ$.
12. Solve $5 \cos^2 \theta = 1$ for $-360^\circ < \theta < 0^\circ$.

To spot a quadratic in disguise you are looking for an equation where the power on one of the variables is twice that on the other. For example

$$(\text{whatever})^{18} + 4(\text{whatever})^9 - 5.$$

This can then factorise to

$$((\text{whatever})^9 + 5)((\text{whatever})^9 - 1),$$

or you can complete the square to

$$((\text{whatever})^9 + 2)^2 - 9.$$

Solve the following:

- | | |
|---|---|
| 1. $x^4 - 13x^2 + 36 = 0.$ | 18. $t = 4\sqrt{t} + 1.$ |
| 2. $x^4 - 15x^2 - 16 = 0.$ | 19. $2^{2x} + 8 = 9 \times 2^x.$ |
| 3. $x^4 + 5x^2 + 6 = 0.$ | 20. $2\sqrt{x} + \frac{9}{\sqrt{x}} = 9.$ |
| 4. $x^6 + 7x^3 = 8.$ | 21. $\left(\frac{1}{x}\right)^2 + 1 = 8\left(\frac{1}{x}\right).$ [Do this question in two ways.] |
| 5. $2x^4 = x^2 + 1.$ | 22. $2(\cos \theta)^2 = 8 \cos \theta + 21.$ |
| 6. $x + 3 = 4\sqrt{x}.$ | 23. $x^8 + 4x^4 = 3.$ |
| 7. $12x^4 = 2x^2 + 4.$ | 24. $2^{2x} + 1 = 2^{x+1}.$ |
| 8. $2(\sin x)^2 + \sin x - 1 = 0$ in range $0 < x < 360.$ | 25. $2^{2x} + 128 = 3 \times 2^{x+3}.$ |
| 9. $\frac{4x^4 + 144}{73} = x^2.$ | 26. $81 + 3^{2x+1} = 4 \times 3^{x+2}.$ |
| 10. $2(\cos x)^2 + (\cos x) = 6.$ | 27. $a^{2x} + a^4 = a^{x+1} + a^{x+3}.$ |
| 11. $x = 2\sqrt{x} + 3.$ | Only attempt the following if you have studied logarithms. |
| 12. $6x^{2/3} + 5x^{1/3} - 4 = 0.$ | 28. $2^{2x} - 13 \times 2^x + 42 = 0.$ |
| 13. $(x^2 - 4x + 1)^2 + (x^2 - 4x + 1) - 12 = 0.$ | 29. $4^{2x} - 9 \times 4^x + 14 = 0.$ |
| 14. $2\theta + 15 = 11\sqrt{\theta}.$ | 30. $3^{2x} + 10 = 7 \times 3^x.$ |
| 15. $x^2 + \frac{72}{x^2} = 17.$ | 31. $2^{2x} - 5 \times 2^{x+1} + 25 = 0.$ |
| 16. $\sqrt{z} - 2\sqrt[3]{z} = 3.$ | 32. $3^{2x} - 3^{x+2} + 20 = 0.$ |
| 17. $2^{2x} - 12 \times 2^x + 32 = 0.$ | |

Purposeful Practice

Give answers to 1 decimal place where necessary.

① Solve the following equations for $0^\circ \leq \theta \leq 360^\circ$.

- | | | |
|---------------------------|------------------------------|--|
| (i) $\cos \theta = 0.5$ | (ii) $\tan \theta = 1$ | (iii) $\sin \theta = \frac{\sqrt{3}}{2}$ |
| (iv) $\sin \theta = -0.5$ | (v) $\cos \theta = 0$ | (vi) $\tan \theta = -5$ |
| (vii) $\tan \theta = 0$ | (viii) $\cos \theta = -0.54$ | (ix) $\sin \theta = 1$ |

② Solve the following equations for $-180^\circ \leq \theta \leq 180^\circ$.

- | | | |
|------------------------------|-----------------------------|-------------------------------|
| (i) $3 \cos \theta = 2$ | (ii) $7 \sin \theta = 5$ | (iii) $3 \tan \theta = 8$ |
| (iv) $6 \sin \theta + 5 = 0$ | (v) $5 \cos \theta + 2 = 0$ | (vi) $5 - 9 \tan \theta = 10$ |

PS ③ Solve the following equations for $0^\circ \leq \theta \leq 360^\circ$.

- | | | |
|----------------------------|----------------------------|---------------------------|
| (i) $\sin^2 \theta = 0.75$ | (ii) $\cos^2 \theta = 0.5$ | (iii) $\tan^2 \theta = 1$ |
|----------------------------|----------------------------|---------------------------|

PS ④ (i) Factorise $2x^2 + x - 1$

(ii) Hence solve $2x^2 + x - 1 = 0$

(iii) Use your results to solve these equations for $-360^\circ \leq \theta \leq 360^\circ$.

- (a) $2 \sin^2 \theta + \sin \theta - 1 = 0$
- (b) $2 \cos^2 \theta + \cos \theta - 1 = 0$
- (c) $2 \tan^2 \theta + \tan \theta - 1 = 0$

← $\sin^2 \theta$ is alternative notation for $(\sin \theta)^2$

PS ⑤ Solve the following equations for $-180^\circ \leq x \leq 180^\circ$.

- | | |
|---------------------------------------|--------------------------------|
| (i) $\tan^2 x - 3 \tan x = 0$ | (ii) $1 - 2 \sin^2 x = 0$ |
| (iii) $3 \cos^2 x + 2 \cos x - 1 = 0$ | (iv) $2 \sin^2 x = \sin x + 1$ |

PS ⑥ Do not use a calculator in this question.

Solve the following equations for $-360^\circ < x < 360^\circ$.

- | | |
|---------------------------------|-----------------------------|
| (i) $\tan x = \sqrt{3}$ | (ii) $2 \sin x = 1$ |
| (iii) $\sqrt{2} \cos x - 1 = 0$ | (iv) $2 \sin x = \sqrt{3}$ |
| (v) $\tan^2 x - \tan x = 0$ | (vi) $4 \cos x = \sqrt{12}$ |

PS ⑦ Solve $(\cos \theta - 1)(\cos \theta + 2)(2 \cos \theta - 1) = 0$ for $0^\circ \leq \theta \leq 360^\circ$.

PS ⑧ (i) Given that $f(x) = 2x^3 - x^2 - 3x - 1$, calculate $f\left(-\frac{1}{2}\right)$.

(ii) Hence solve $2 \sin^3 \theta - \sin^2 \theta - 3 \sin \theta - 1 = 0$ for $-180^\circ \leq \theta \leq 180^\circ$.

Purposeful Practice

- ① For each of the equations (i)–(v):
- (a) use the identity $\sin^2 \theta + \cos^2 \theta \equiv 1$ to rewrite the equation in a form involving only one trigonometric function
- (b) factorise, and hence solve, the resulting equation for $0^\circ \leq \theta \leq 360^\circ$.
- (i) $2 \cos^2 \theta + \sin \theta - 1 = 0$ (ii) $\sin^2 \theta + \cos \theta + 1 = 0$
- (iii) $2 \sin^2 \theta - \cos \theta - 1 = 0$ (iv) $\cos^2 \theta + \sin \theta = 1$
- (v) $1 + \sin \theta - 2 \cos^2 \theta = 0$
- ② For each of the equations (i)–(iii):
- (a) use the identity $\sin^2 \theta + \cos^2 \theta \equiv 1$ to rewrite the equation in a form involving only one trigonometric function
- (b) use the quadratic formula to solve the resulting equation for $0^\circ \leq \theta \leq 180^\circ$.
- (i) $\sin^2 \theta - 2 \cos \theta + 1 = 0$
- (ii) $\cos^2 \theta - \sin \theta = 0$
- (iii) $\sin^2 \theta - 3 \cos \theta = 0$
- ③ (i) Use the identity
- $$\tan \theta \equiv \frac{\sin \theta}{\cos \theta}$$
- to rewrite the equation $\sin \theta = 2 \cos \theta$ in terms of $\tan \theta$.
- (ii) Hence solve the equation $\sin \theta = 2 \cos \theta$ for $0^\circ \leq \theta \leq 180^\circ$.

- ④ Use the identity
- $$\tan \theta \equiv \frac{\sin \theta}{\cos \theta}$$
- to solve the following equations for $0^\circ \leq \theta \leq 360^\circ$.

- (i) $2 \sin \theta + \cos \theta = 0$
- (ii) $\sqrt{3} \tan \theta = 2 \sin \theta$
- (iii) $4 \cos \theta \tan \theta = 1$

- PS ⑤ Write the following in terms of $\sin x$.

- (i) $\cos^2 x \tan^2 x$ (ii) $\tan x \cos^3 x$ (iii) $\cos x (2 \cos x - 3 \tan x)$

- PS ⑥ Show that $(3 \sin x)(\sin x + 2) - 3(2 \sin x - \cos^2 x)$ simplifies to an integer.

- PS ⑦ Prove the following identities.

- (i) $\tan x \sqrt{1 - \sin^2 x} \equiv \sin x$
- (ii) $\frac{1 - \cos^2 x}{1 - \sin^2 x} \equiv \tan^2 x$
- (iii) $(1 + \sin x)(1 - \sin x) \equiv \cos^2 x$
- (iv) $\frac{2 \sin x \cos x}{\tan x} \equiv 2 - 2 \sin^2 x$

- PS ⑧ Solve $5 \sin x (\sin x + \cos x) = 3$ for $0^\circ < x < 360^\circ$.

Hint: Replace 3 with 3×1 and then replace the 1 with $\sin^2 x + \cos^2 x$.

Solving Trig Equations with... **The Factor Theorem**

Solve each equation in the given region:

Round answers to 1 decimal place, where appropriate.

- 1) $6 \sin^3 x - 5 \sin^2 x - 3 \sin x + 2 = 0$, for $0^\circ \leq x < 360^\circ$,
- 2) $12 \cos^4 x - \cos^3 x - 18 \cos^2 x + \cos x + 6 = 0$, for $-180^\circ \leq x < 180^\circ$,
- 3) $6 \tan^5 x + 35 \tan^4 x + 62 \tan^3 x + 35 \tan^2 x + 6 \tan x = 0$, for $0^\circ \leq x < 180^\circ$.

- 4) $2 \cos^3 x + 3 \sin^2 x - 8 \cos x - 6 = 0$, for $0^\circ \leq x < 720^\circ$,
- 5) $-3 \sin(x) \cos^2 x + 11 \sin^2 x - 16 \sin x + 5 = 0$, for $-360^\circ \leq x < 360^\circ$,
- 6) $\tan(x) \sin^2 x - 3 \sin^2 x - 10 \sin(x) \cos x + 24 \cos^2 x = 0$, for $-360^\circ \leq x < 0^\circ$.

- 7) $6 \sin^4 2x - 5 \sin^3 2x - 14 \sin^2 2x - \sin 2x + 2 = 0$, for $0^\circ \leq x < 180^\circ$,
- 8) $5 \cos^5 3x - 19 \cos^4 3x - 9 \cos^3 3x + 79 \cos^2 3x - 44 \cos 3x - 12 = 0$, for $0^\circ \leq x < 120^\circ$,
- 9) $\tan^4(4x + 5) - 27 \tan^2(4x + 5) - 14 \tan(4x + 5) + 120 = 0$, for $0^\circ \leq x < 90^\circ$.