KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 11

2025 Level 2 Further Mathematics 2026

Unit 29 Booklet

HGS Maths

Tasks

Dr Frost Course

Name: ____________________________________

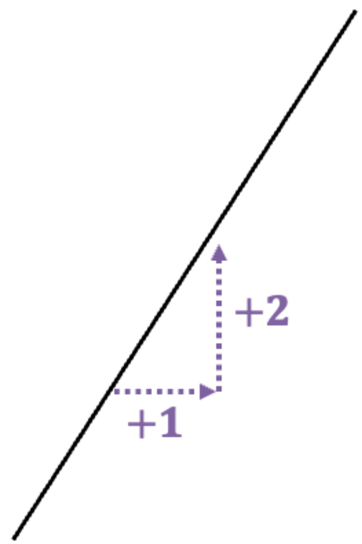
Class: ____________________________________

Contents

1 Differentiation

Basic Differentiation

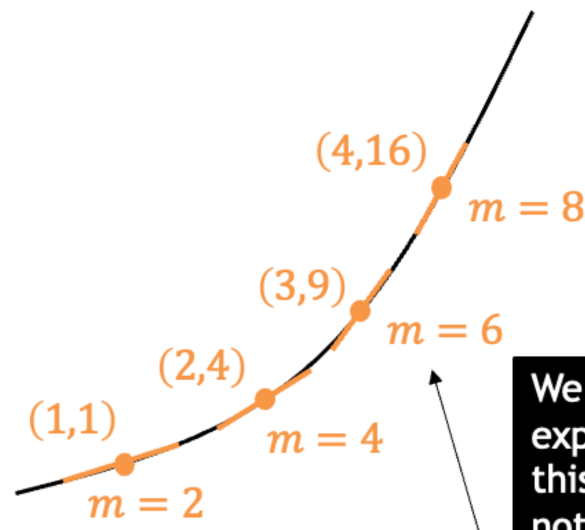
The Aim in Differentiation



Equation: $y = 2x + 1$

Gradient: $m = 2$

As mentioned, we can use a single value to represent the gradient across the line.



We will explore this notation later.

Equation: $y = x^2$

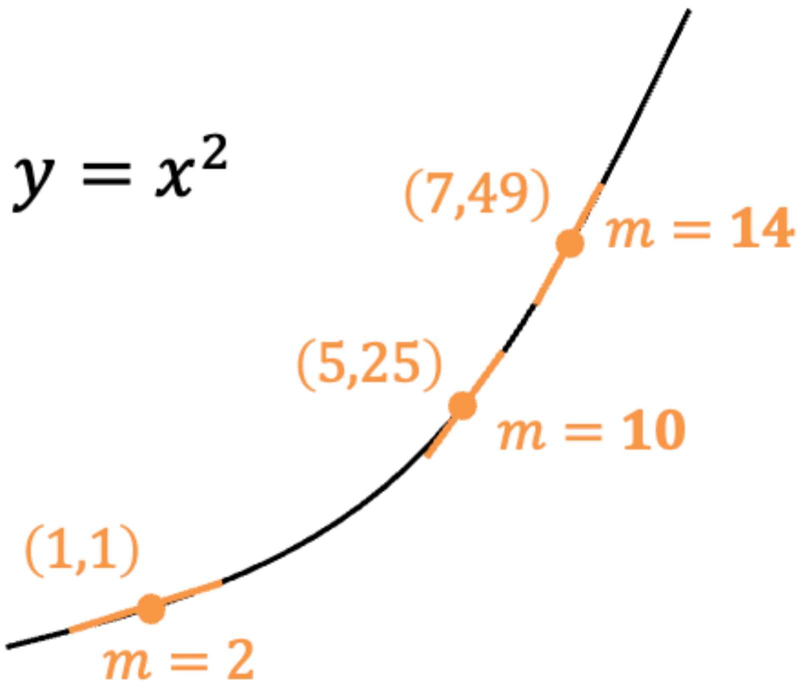
Gradient function: $\frac{dy}{dx} = 2x$

Can you think of an expression in terms of x for the gradient?

For a curve, we want a <u>**function**</u> that allows us to calculate the gradient for a specific point on the curve. This is known as <u>**differentiation**</u>.

Understanding the Gradient Function

$$\frac{dy}{dx} = 2x$$



This means that the gradient of (any tangent to) the curve at a point (x, y) will be $2x$, i.e. twice the x value.

Differentiating Powers of x

If $y = ax^n$ then $\frac{dy}{dx} = nax^{n-1}$
(where a, n are constants)
i.e. multiply by the exponent and reduce the exponent by 1

Misconception

Why would it be incorrect to say that $y = 2^x$ differentiates to $\frac{dy}{dx} = x\,2^{x-1}$?

The rule only works when the base is x and the power is a constant. Neither is true here! Note that x^n is "a power of x" whereas 2^x is an exponential term and therefore differentiate differently.

Specification
Spec Note: For IGCSEFM, the powers will always be positive integers, not fractional.

Notation for Gradient

$$\frac{d}{dx}(x^3) = 3x^2$$

We can also conveniently show a function being differentiated in a single equation.

$\frac{d}{dx}$ means **"differentiate the following with respect to x"**.

$$\frac{d}{dr}(r^5) = 5r^4$$

We needn't differentiate in terms of x. The variable at the bottom of the $\frac{d}{d\square}$ indicates what variable we are differentiating with respect to.

Worked Example	Your Turn
Differentiate with respect to x: a) x^4 b) $5x^4$ c) $4x - 5$ d) $\frac{1}{x^3}$ e) $\frac{6}{5x^3}$	Differentiate with respect to x: a) x^5 b) $-3x^5$ c) $6x - 7$ d) $\frac{1}{x^4}$ e) $\frac{7}{8x^4}$

Notation for Gradient

Original function		Gradient function	
y	→	$\frac{dy}{dx}$	This is known as **Leibniz notation**
$f(x)$	→	$f'(x)$	This is known as **Lagrange notation**
y	→	$\dot{y}$	This is known as **Newton's notation**, which is used in Mechanics/Physics. It specifically means the rate of change with respect to **time**.

Worked Example	Your Turn
Differentiate with respect to x: a) $y = 4x^3 + 3x^2 + 2x + 1$ b) $y = (5x - 3)^2$ c) $f(x) = x^3(x + 2)$	Differentiate with respect to x: a) $y = 5x^4 - 2x^7 + 12345 - x^5$ b) $y = (3x - 5)^2$ c) $f(x) = x^2(x - 3)$

Dr Frost 512a and 512b

Worked Example	Your Turn
a) A curve has equation $y = 2x^3 + kx^2 - 6x - 4$, where k is a constant. Find $\frac{dy}{dx}$ b) A curve has equation $y = f(x)$, where $f(x) = x^3 + bx^2 - 6x + 6$ Given that $f'(-1) = -11$ Find the value of b	a) A curve has equation $y = px^2 + 5x - 8$, where p is a constant. Find $\frac{dy}{dx}$ b) A curve has equation $y = f(x)$, where $f(x) = rx^2 - 2x + 6$ Given that $f'(4) = 38$ Find the value of r

Worked Example	Your Turn
Find an expression for the rate of change of y with respect to x: $y = (x - 3)(x - 4)^2$	Find an expression for the rate of change of y with respect to x: $y = (x + 1)(x + 2)^2$

Dr Frost 512i

Worked Example	Your Turn
Differentiate with respect to x: a) $f(x) = \frac{(3x-2)^2}{5x}$ b) $y = \frac{3x^4(2x^5-11x)}{x^2}$	Differentiate with respect to x: a) $f(x) = \frac{(2x+3)^2}{5x}$ b) $f(x) = \frac{2x^2(3x^3-7x)}{x}$

Worked Example	Your Turn
a) Find the gradient of the curve: $y = 2x^3 - x + 5$ at $(-1, 4)$ b) Find the coordinates of the point(s) where the gradient is 4: $y = 5x^2 - x + 7$	a) Find the gradient of the curve: $y = 3x^2 - 2x + 1$ at $(-2, 17)$ b) Find the coordinates of the point(s) where the gradient is 3: $y = 3x^2 - 9x + 7$

Worked Example	Your Turn
A curve has gradient function $\frac{dy}{dx} = 3x^2 + 7$ Work out the values of x for which the rate of change of y with respect to x is 55	A curve has gradient function $\frac{dy}{dx} = 5x^2 - 7$ Work out the values of x for which the rate of change of y with respect to x is 38

Worked Example

Let $f(x) = 4x^2 - 8x + 3$

a) Find the gradient of $y = f(x)$ at the point $\left(\frac{1}{2}, 0\right)$

b) Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 8

c) Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 4x - 5$

Your Turn

Let $f(x) = x^2 - 4x + 2$

a) Find the gradient of $y = f(x)$ at the point $(1, -1)$

b) Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 5

c) Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 2 - x$

Worked Example	Your Turn
Find the coordinates of the point(s) where the gradient is 10: $y = x^3 + 6x^2 - 26x + 7$	Find the coordinates of the point(s) where the gradient is 2: $y = x^3 - 3x^2 - 7x + 8$

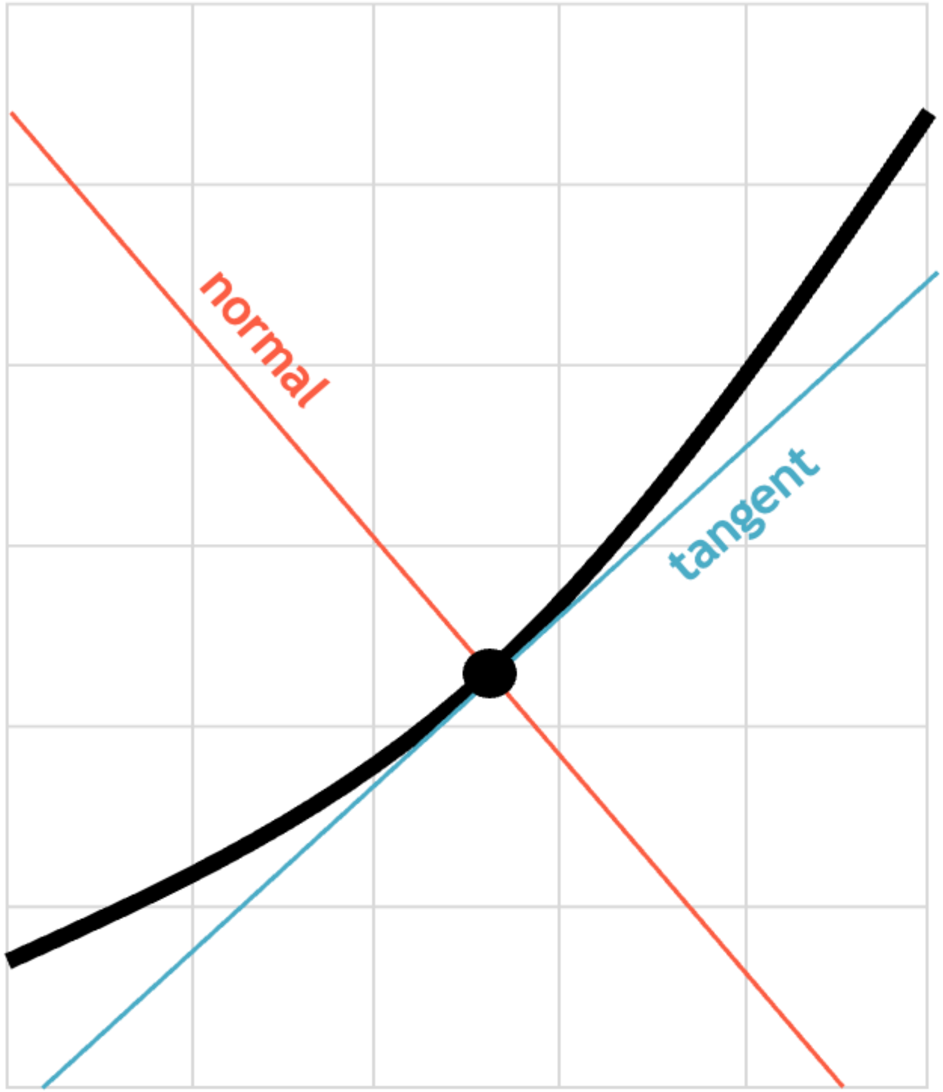
Worked Example	Your Turn
$y = 3x^2 + bx$ The rate of change of y with respect to x when $x = 4$ is triple the rate of change of y with respect to x when $x = -2$ Work out the value of b	$y = 2x^3 + ax$ The rate of change of y with respect to x when $x = 2$ is twice the rate of change of y with respect to x when $x = -1$ Work out the value of a

Fill in the Gaps

Equation	Gradient Function	Point P	Gradient at P
$y = x^2$	$\frac{dy}{dx} = 2x$	$(2, 4)$	4
$y = x^3 + x$		$(1, 2)$	
$y = 6x - x^2$		$(4, 8)$	
$y = x^3 - 3x^2 + 4x$		$(-1, 0)$	
$y = 5x^2 - 7x + 1$		$(-2, 36)$	
$y = (2x + 5)(x - 3)$		$(3, 0)$	
$y = 3x(x - 1)^2$		$(-1, -12)$	
$y = \frac{1}{x^2}$		$\left(2, \frac{1}{4}\right)$	
$y = \frac{x^4 - 5x^3}{x}$		$(1, -4)$	
$y = \frac{2x^3 + x}{x^2}$		$\left(3, \frac{19}{3}\right)$	
$y = 10 - 2x - x^2$			-10
$y = x^4 + 3$			32
$y = (x + 4)(3x - 5)$			1
$y = x^2 + \frac{54}{x}$			0
	$\frac{dy}{dx} = 3x^2 + 6x - 1$	$(1, 3)$	

Tangents and Normals

Tangents and Normals

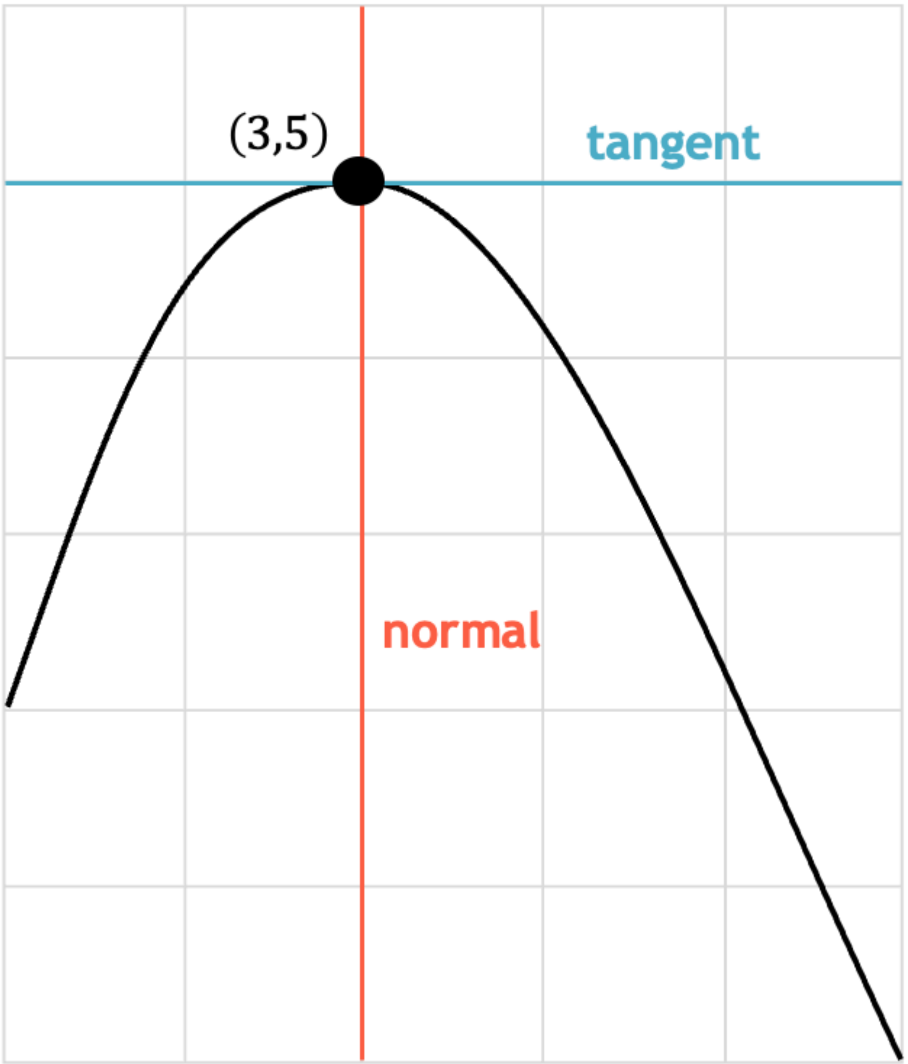


We saw that a tangent to the curve is a line that **touches** the curve.

The gradient function $\frac{dy}{dx}$ **gives us the gradient of this tangent**.

A normal to the curve is a line **perpendicular to the tangent**, i.e. comes out of the curve.

Equations of Tangents and Normals at Stationary Points



Recall that a **stationary point** is a point where the gradient is 0.

What is the equation of the tangent at the point (3,5)?

By observation:

$$y = 5$$

What is the equation of the normal at the point (3,5)?

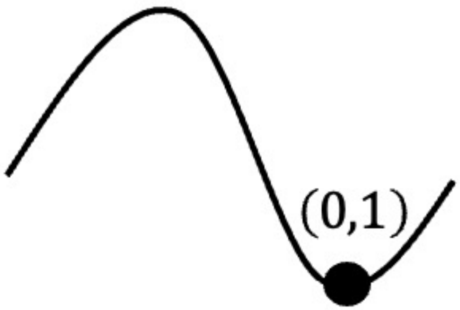
$$x = 3$$

Dr Frost 520a

Fluency Practice

Determine the equations of the tangents and normal at the following stationary points:

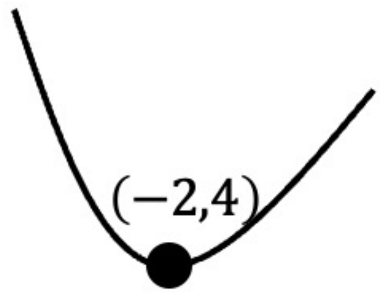
a



Equation of:

Tangent	?
Normal	?

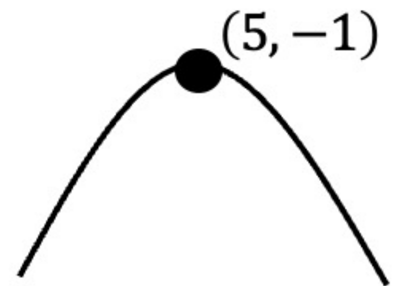
b



Equation of:

Normal	?
Tangent	?

c



Equation of:

Tangent	?
Normal	?

Worked Example	Your Turn
a) Find the equation of the tangent to the curve $y = x^4$ when $x = 2$ b) Find the equation of the normal to the curve $y = x^4$ when $x = 2$	a) Find the equation of the tangent to the curve $y = x^3$ when $x = 2$ b) Find the equation of the normal to the curve $y = x^3$ when $x = 2$

Worked Example	Your Turn
Find the equation of the tangent to the curve with equation $y = x^3 - 5x^2 - 3x + 2$ at the point $(5, -13)$	Find the equation of the normal to the curve with equation $y = x^3 - 3x^2 + 2x - 1$ at the point $(3, 5)$

Dr Frost 520c and 520d

Worked Example	Your Turn
The tangent to the curve $y = \frac{1}{4}x^3 + x^2 - 3x$ at the point P is parallel to the line with equation $y = -\frac{17}{4}x + 2$. Find the two possible values for the x-coordinate of the point P.	The tangent to the curve $y = \frac{1}{5}x^3 - x^2 - x$ at the point P is parallel to the line with equation $y = 4x - 9$. Find the two possible values for the x-coordinate of the point P.

Dr Frost 520f and 520e

Worked Example	Your Turn
The point P with x-coordinate $\frac{1}{4}$ lies on the curve with equation $y = 2x^2$. The normal to the curve at P intersects the curve at points P and Q. Find the coordinates of Q	The point P with x-coordinate $\frac{1}{2}$ lies on the curve with equation $y = 4x^2$. The normal to the curve at P intersects the curve at points P and Q. Find the coordinates of Q

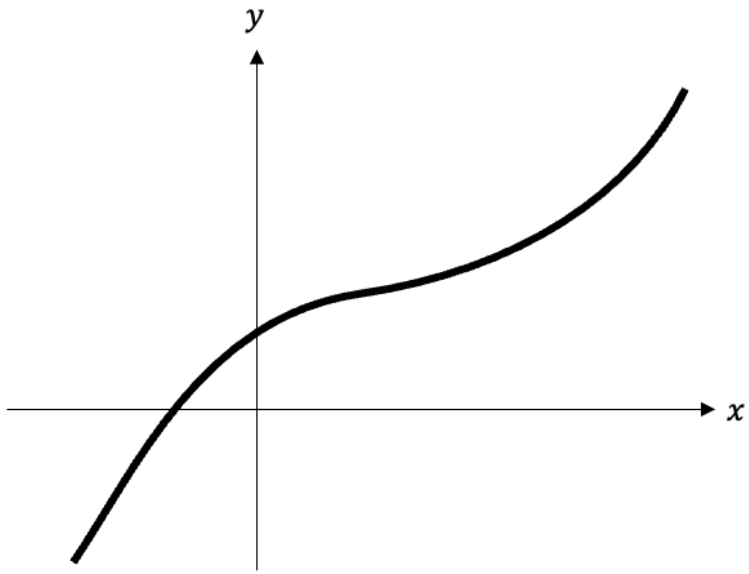
Worked Example	Your Turn
The curve $y = 4x^3 - 7$ intersects the y-axis at C. The tangent to the curve at $P(3, 101)$ intersects the y-axis at D Work out the length of CD	The curve $y = 2x^3 - 5$ intersects the y-axis at C. The tangent to the curve at $P(2, 11)$ intersects the y-axis at D Work out the length of CD

Dr Frost 520h

Worked Example	Your Turn
Point B lies on the curve $y = x^3 - 5x + 4$ The x-coordinate of B is -8 Show that the equation of the normal to the curve at B is $187y + x = -87524$	Point A lies on the curve $y = x^2 + 5x + 8$ The x-coordinate of A is -4 Show that the equation of the normal to the curve at A is $3y - x = 16$

Increasing and Decreasing Functions

Increasing and Decreasing Functions

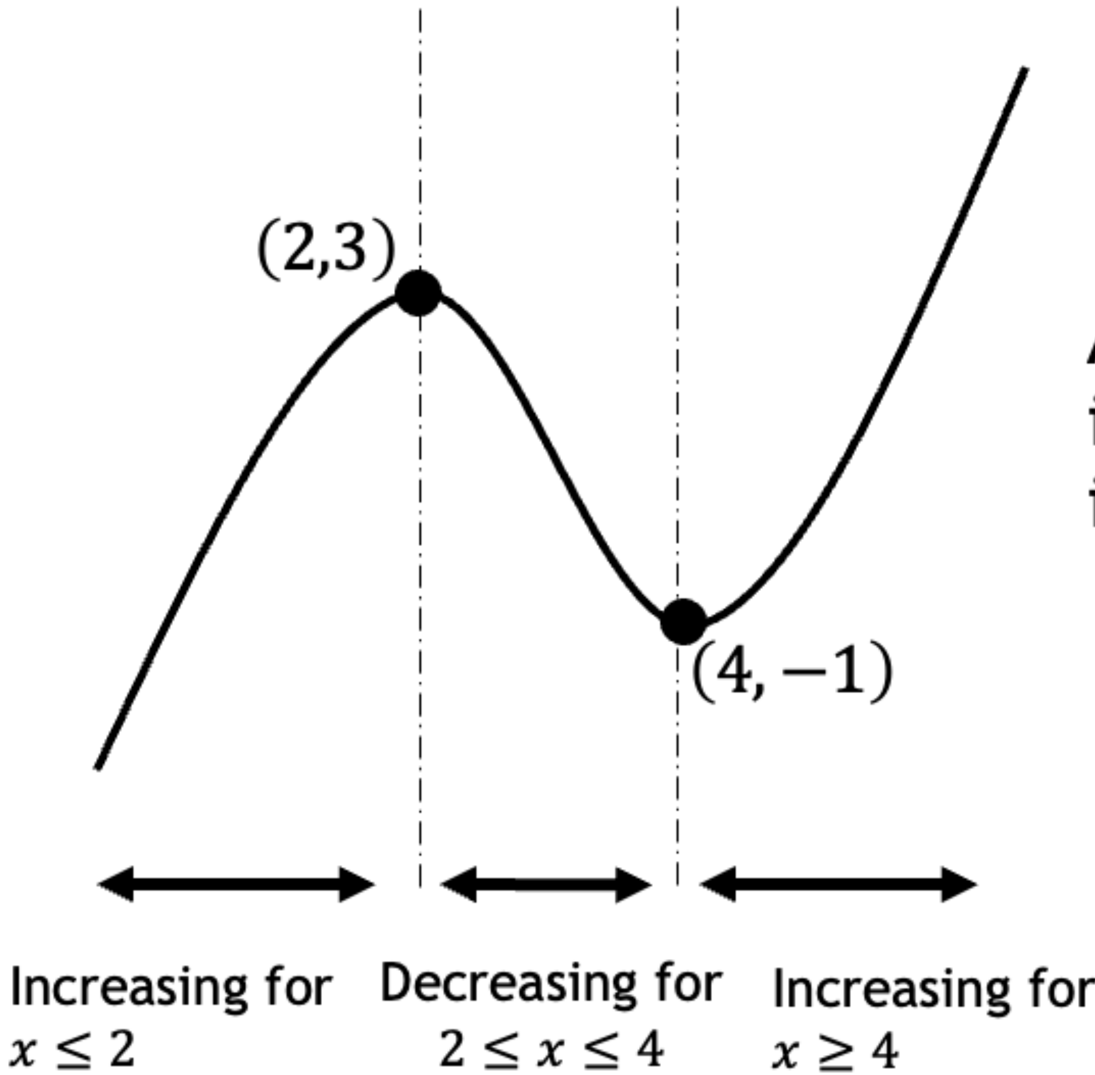


What do you think it means for a function to be an '**increasing function**'?

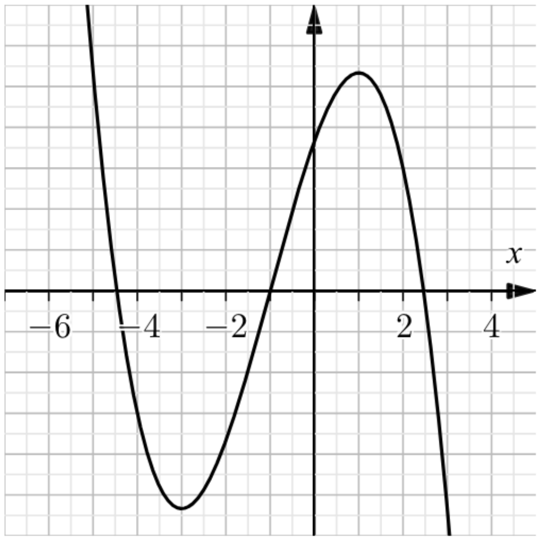
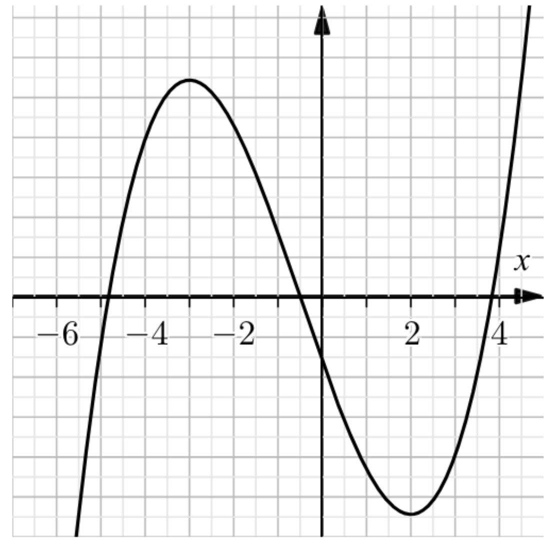
An increasing function is one whose gradient is always at least* 0.
$f'(x) \geq 0$ for all x.

* It would be '**strictly increasing**' if $f(x) > 0$ for all x, i.e. is not allowed to go horizontal. Some exam boards consider an 'increasing function' to be $f(x) > 0$, so beware. Mark schemes may allow both $>$ and $\geq$.

Increasing and Decreasing Functions



A function can also be increasing and decreasing in certain intervals.

Worked Example	Your Turn
A sketch of the curve $y = f(x)$ is shown below.  Using the graph, find the interval in which $f(x)$ is decreasing.	A sketch of the curve $y = f(x)$ is shown below.  Using the graph, find the interval in which $f(x)$ is increasing.

Dr Frost 516a

Worked Example	Your Turn
Find the values of x for which the function $f(x) = x^2 - 5x + 8$ is increasing	Find the values of x for which the function $f(x) = x^2 + 8x - 5$ is decreasing

Dr Frost 516b

Worked Example	Your Turn
A cubic function is given by the equation $f(x) = -3x^3 - 6x^2 - 3x + 5$ Determine whether is increasing, decreasing, or neither increasing nor decreasing, when $x = -2$	A cubic function is given by the equation $f(x) = -x^3 + 8x^2 + 35x + 4$ Determine whether is increasing, decreasing, or neither increasing nor decreasing, when $x = 7$

Dr Frost 516c

Worked Example	Your Turn
Find the interval(s) on which the function $f(x) = x^3 - 6x^2 - 135x + 1$ is increasing	Find the interval(s) on which the function $f(x) = x^3 + 6x^2 - 135x - 2$ is increasing

Worked Example	Your Turn
Find the interval on which the function $f(x) = x^3 - 3x^2 - 9x - 10$ is decreasing	Find the interval on which the function $f(x) = x^3 + 3x^2 - 9x + 5$ is decreasing

Dr Frost 516d

Worked Example	Your Turn
Show that the function $f(x) = x^3 + 26x - 1$ is increasing for all real values of x	Show that the function $f(x) = x^3 + 16x - 2$ is increasing for all real values of x

Tip
Tip: To show a quadratic is always positive, complete the square, then indicate the squared term is always at least 0.

Worked Example	Your Turn
Show that the function $f(x) = x^3 - 3x^2 + 8x - 5$ is increasing for all real values of x	Show that the function $f(x) = x^3 + 6x^2 + 21x + 2$ is increasing for all real values of x

Worked Example	Your Turn
Show that the function $5 - x(4x^2 + 3)$ is decreasing for all real values of x	Show that the function $3 + 4x(-x^2 - 5)$ is decreasing for all real values of x

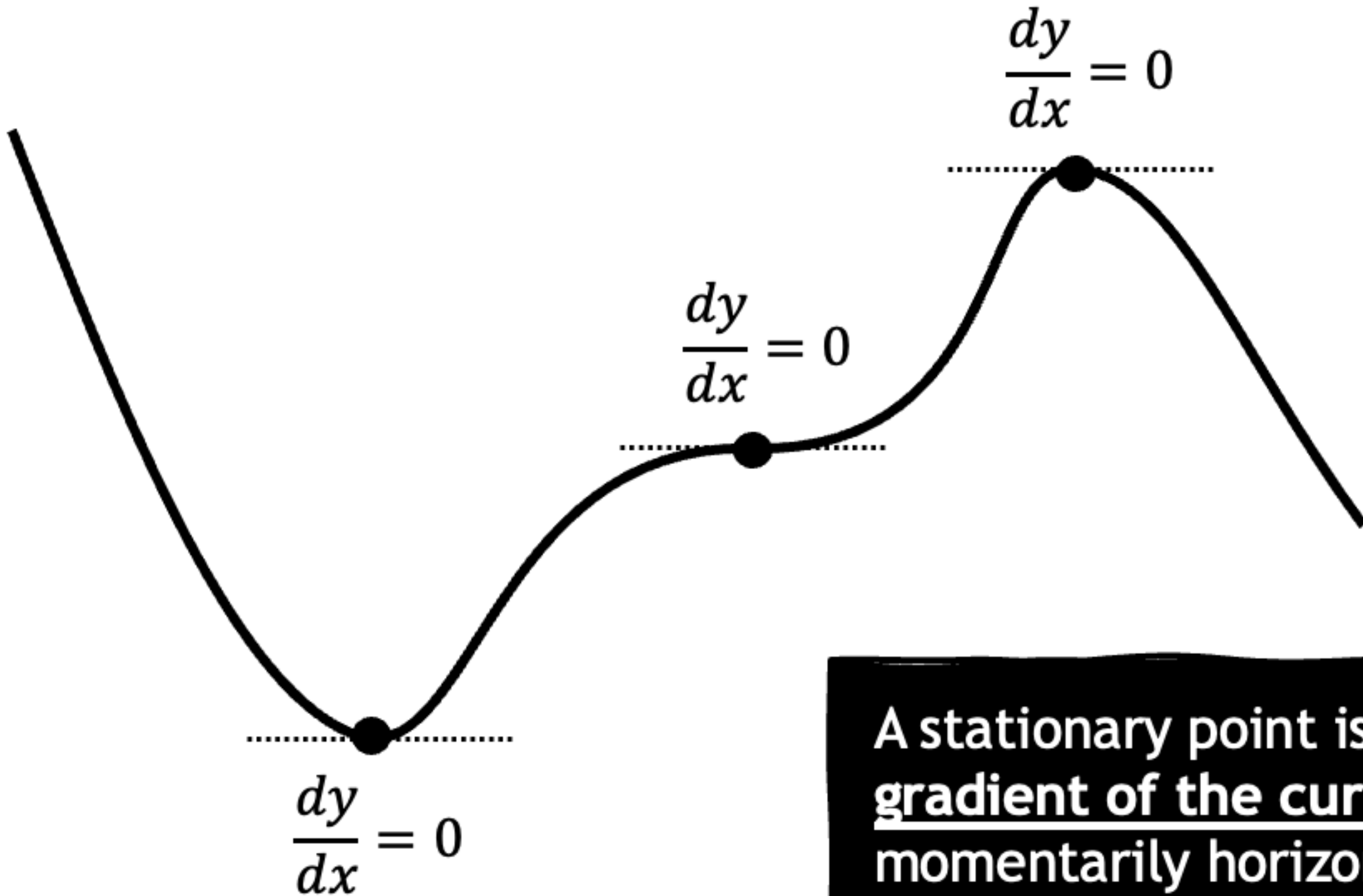
Dr Frost 516e

Worked Example	Your Turn
A function is given by the equation $f(x) = 2x + \frac{32}{x} - 19, x \neq 0$ Find the exact interval on which is increasing.	A function is given by the equation $f(x) = -2x - \frac{49}{x} - 4, x \neq 0$ Find the exact interval on which is decreasing.

Dr Frost 516f

Stationary Points

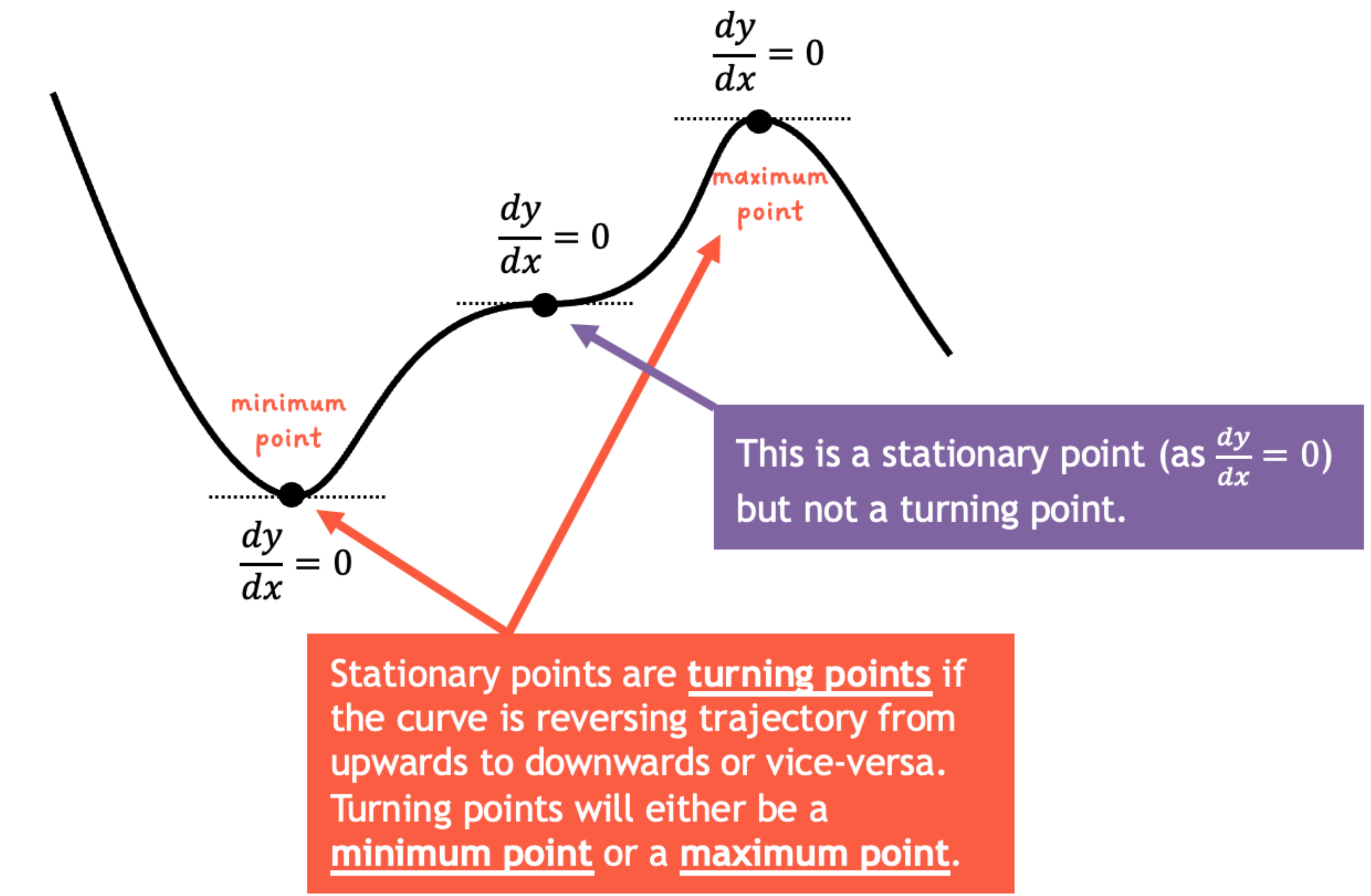
What is a Stationary Point?



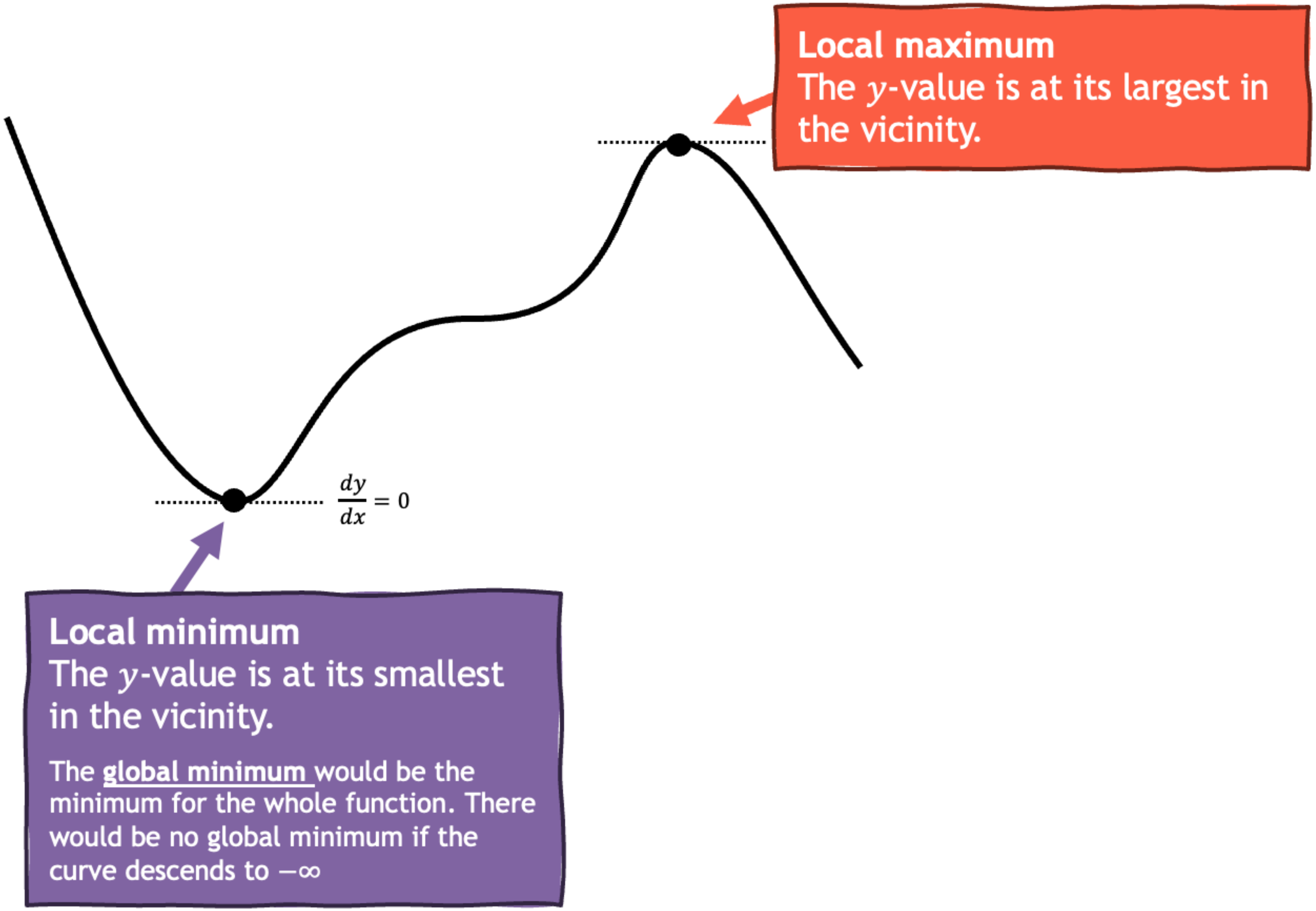
A stationary point is where the **gradient of the curve is 0**, i.e. momentarily horizontal.

$$\frac{dy}{dx} = 0$$

Stationary Points vs Turning Points



Stationary Points



Worked Example	Your Turn
Find the least value of $f(x) = x^2 + 6x - 9$	Find the least value of $f(x) = x^2 - 4x + 9$

Dr Frost 517a

Worked Example	Your Turn
Find the coordinates of the turning/stationary point(s) of the curves by differentiation: $y = 2x^3 + 6x^2 - 4$	Find the coordinates of the turning/stationary point(s) of the curves by differentiation: $y = x^3 + 3x^2 - 4$

Worked Example	Your Turn
Find the stationary points on the curve $y = \frac{5}{3}x^3 - 80x$	Find the stationary points on the curve $y = x^3 - 12x$

Worked Example	Your Turn
Find the coordinates of the turning points of $y = x^3 - 6x^2 - 15x$	Find the coordinates of the turning points of $y = x^3 + 6x^2 - 135x$

Worked Example	Your Turn
Find the coordinates of the turning/stationary point(s) of the curves by differentiation: $y = \frac{2}{3}x^3 - 3.5x^2 + 3x + 5$	Find the coordinates of the turning/stationary point(s) of the curves by differentiation: $y = x^3 + \frac{1}{2}x^2 - 2x + 4$

Dr Frost 517b

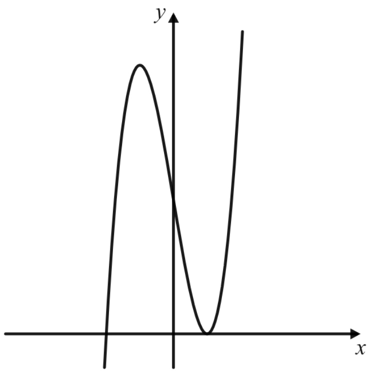
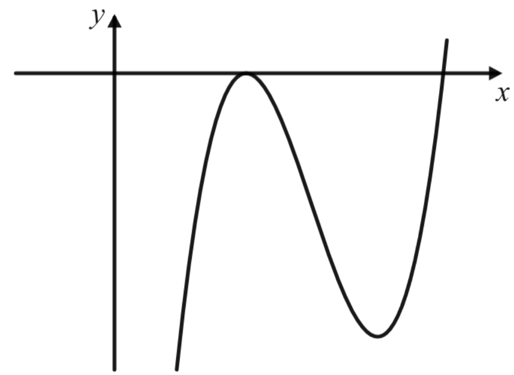
Worked Example	Your Turn
A graph has equation $y = 4 - \frac{27}{x} - 3x$ Determine the x-coordinates of the stationary points.	A graph has equation $y = 5 - \frac{2}{x} - 2x$ Determine the x-coordinates of the stationary points.

Dr Frost 517f

Worked Example	Your Turn
$y = 4ax^3 + \frac{3}{x}$ y has a minimum when $x = \frac{1}{3}$ Work out the value of a	$y = 8ax^3 + \frac{6}{x}$ y has a minimum when $x = \frac{1}{2}$ Work out the value of a

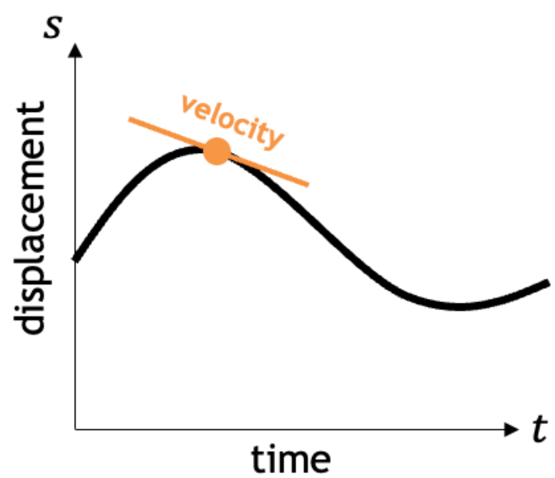
Worked Example	Your Turn
The curve $y = x^3 + ax + b$ has a stationary point at $(-2, 3)$ Work out the values of a and b	The curve $y = x^3 + ax^2 + b$ has a stationary point at $(3, -8)$ Work out the values of a and b

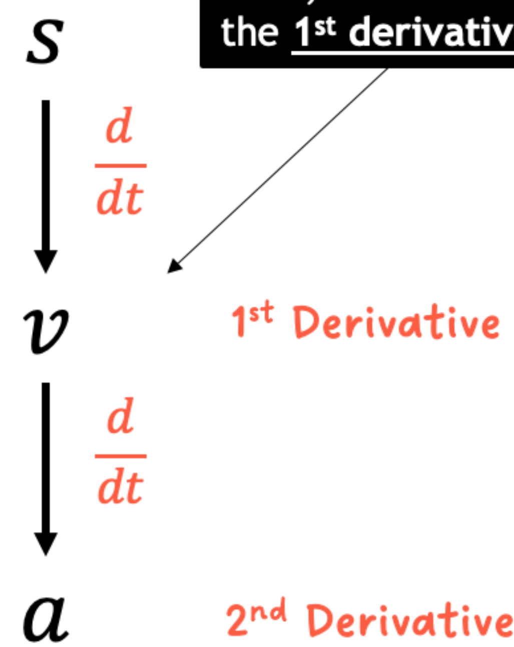
Dr Frost 517j

Worked Example	Your Turn
The curve sketched below has equation $y = 2x^3 + ax^2 - 6x + b$  The x-axis is a tangent to the curve at its minimum point where $x = 1$ Determine the coordinate of the curve's maximum point.	The curve sketched below has equation $y = 2x^3 - 18x^2 + ax + b$  The x-axis is a tangent to the curve at its maximum point where $x = 2$ Determine the coordinate of the curve's minimum point.

Dr Frost 517k

What is the Second Derivative?





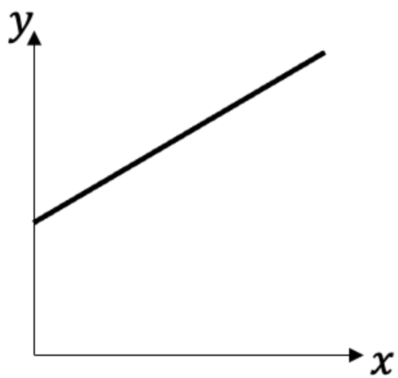
If we've differentiated once, we call this the **1st derivative**.

We previously saw that velocity was the **gradient** of a displacement-time graph, i.e. velocity is the **rate of change** of displacement.

But similarly, the gradient of a velocity-time graph gives us acceleration.
So velocity is the 1st derivative of displacement, and acceleration is the **2nd derivative** of displacement.

What is the Second Derivative?

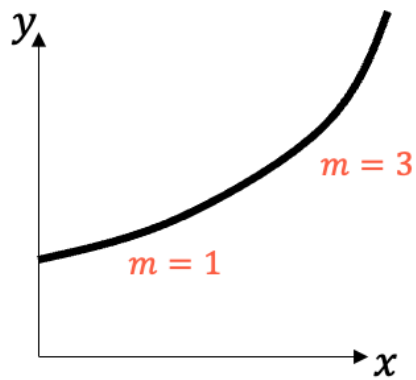
The 2nd derivative also gives us a way of understanding how **gradient is changing**.



In this graph, the **y value is increasing**, so the **gradient of y is positive**.

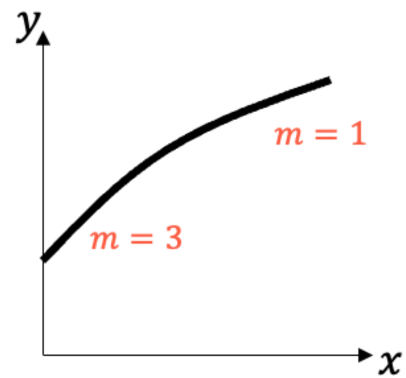
But the gradient is not changing, **so the gradient of the gradient is 0**.

In other words, the **second derivative is 0**.



In this graph, the **gradient is increasing** (i.e. getting gradually steeper), so the **gradient of the gradient is positive**.

In other words, the **second derivative is positive**.



Similarly, the **gradient is decreasing** (even if still positive), so the gradient of the gradient is negative.

In other words, the **second derivative is negative**.

Notation for Gradient

	Original function	1st Derivative	2nd Derivative	
Leibniz Notation	y	$\dfrac{dy}{dx}$	$\dfrac{d^2y}{dx^2}$	
Lagrange Notation	$f(x)$	$f'(x)$	$f''(x)$	
Newton's Notation	y	$\dot{y}$	$\ddot{y}$	This is used in Physics/ Engineering when differentiating **with respect to time**.

Why is the Second Derivative Written This Way?

We understand d to mean "the (infinitesimally small) change in whatever comes after".

$$\frac{d}{dx}(y) = \frac{dy}{dx}$$

$$\frac{d}{dx}\left(\frac{d}{dx}(y)\right) = \frac{d^2y}{dx^2}$$

If we think of the d and dx as if they were quantities, we can see how we get the resulting notation $\frac{d^2y}{dx^2}$ by multiplication.

d^2 is therefore a convenience to mean "consider the change of the change in whatever comes after" and dx^2 (which really means $(dx)^2$) represents "with respect to the change in x, twice!"

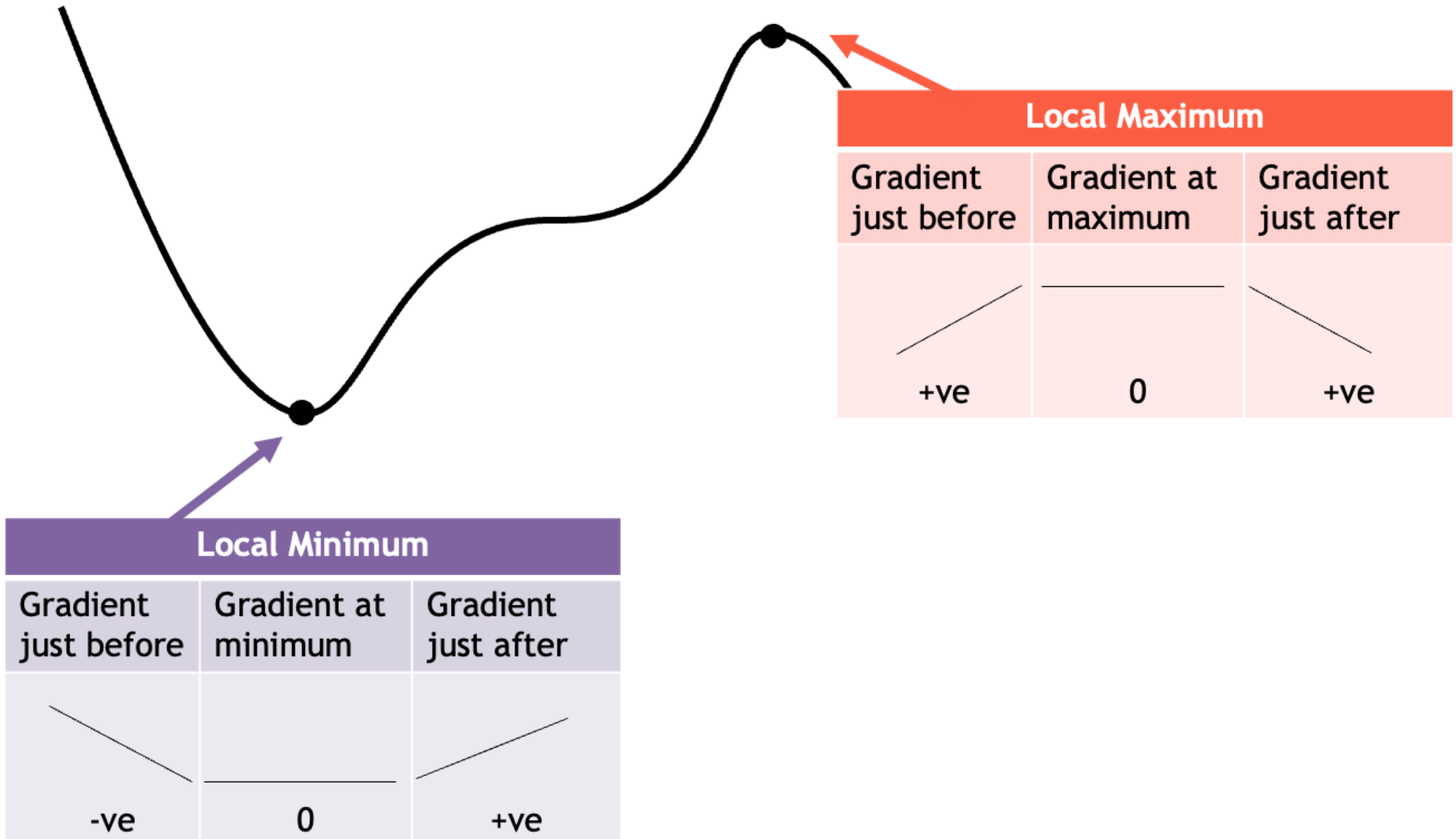
The reason we don't write $\frac{d^2y}{d^2x}$ is that dx within $\frac{d}{dx}$ is considered a single thing.

Worked Example	Your Turn
$y = 5x^3 - 7x^2 - x + 4$ Find the value of $\frac{d^2y}{dx^2}$ when $x = 2$	$y = 2x^3 - 3x^2 + x + 9$ Find the value of $\frac{d^2y}{dx^2}$ when $x = 2$

Dr Frost 515b and 515a

Method 1

Method 1: Look at gradient just before and just after the point.

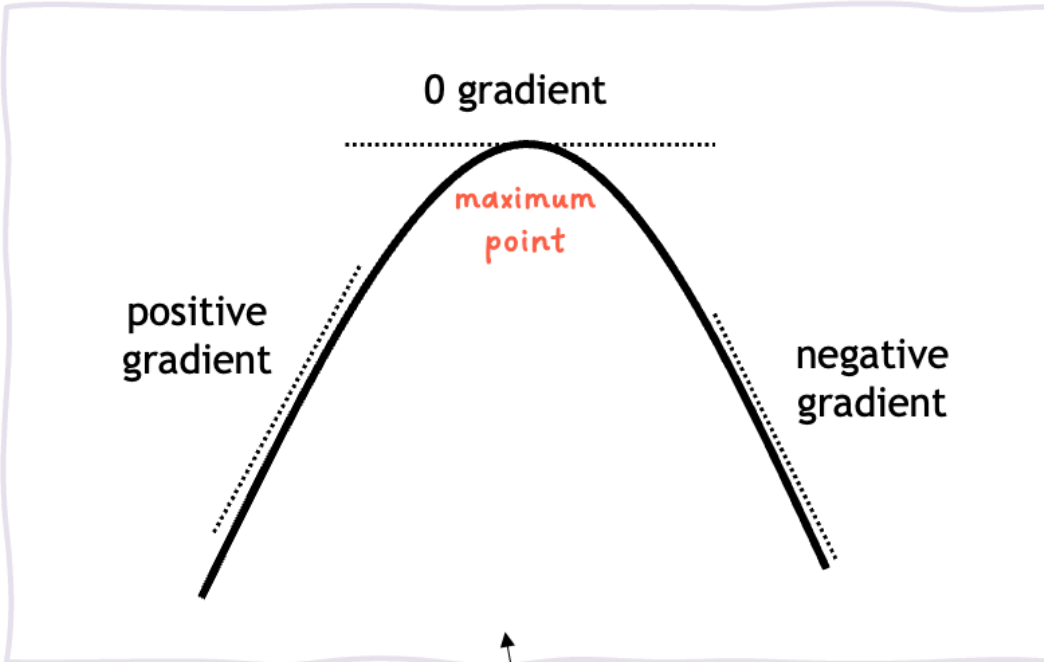


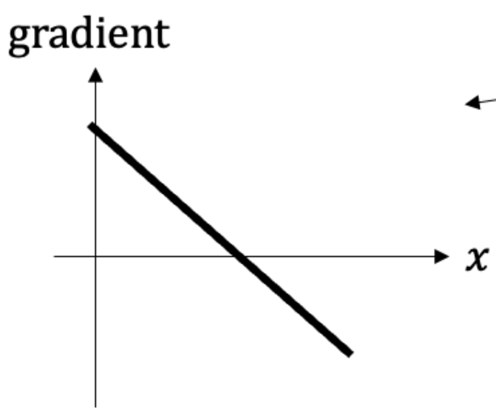
Worked Example	Your Turn
Show that one of the stationary points of the curve with equation $y = x^3 - 3x^2 + 45x$ is $(-3, -159)$, and by testing the gradient of the curve either side of the stationary point, determine whether it is a maximum or a minimum.	Show that one of the stationary points of the curve with equation $y = x^3 + 3x^2 - 45x$ is $(3, -81)$, and by testing the gradient of the curve either side of the stationary point, determine whether it is a maximum or a minimum.

Specification

Many students already use the alternative method of finding $\frac{d^2y}{dx^2}$ to determine the nature of stationary points; and so by including the second differential in this specification, instantaneous rates of change can be more fully discussed; and in this context the rate of change of the gradient function. Stationary points will be restricted to maxima and minima; and consequently finding points of inflection (using $\frac{d^2y}{dx^2} = 0$) will **not** be assessed.

Method 2





Above, **the gradient is decreasing** from positive to negative.

Anything that is decreasing has negative gradient. And it's the gradient that's decreasing, so **the gradient of the gradient is negative**!

We saw that the 'gradient of the gradient' is known as the **second derivative**, written as $\frac{d^2y}{dx^2}$

Therefore, for a maximum point, $\frac{d^2y}{dx^2} < 0$

Method 2

At a stationary point $x = a$:

- If $f''(a) > 0$ the point is a local minimum.
- If $f''(a) < 0$ the point is a local maximum.
- If $f''(a) = 0$ it could be any type of point, so resort to Method 1.

Worked Example	Your Turn
Find the coordinates of the stationary points on the curve with equation $y = 4x^3 + 30x^2 + 48x - 3$ and use the second derivative to determine their nature.	Find the coordinates of the stationary points on the curve with equation $y = 2x^3 - 15x^2 + 24x + 6$ and use the second derivative to determine their nature.

Dr Frost 517d

Fill in the Gaps

Equation of Curve	$\frac{dy}{dx}$	$\frac{dy}{dx} = 0$	x-coordinate	y-coordinate	Maximum or Minimum Point
$y = x^2 - 10x + 2$				$y = -23$	$Minimum$
$y = 3x^2 + 12x + 20$					
$y = 15 - 2x - x^2$					
$y = 3 + 8x - 2x^2$					
$y = x^2 + 12x + \square$				$y = -6$	$Minimum$
$y = x^2 - 9x + \square$				$y = -\frac{21}{4}$	
$y = x^2 - \square x + 15$			$x = 4$		
$y = \square + \square\ x - x^2$			$x = 2$	$y = 10$	

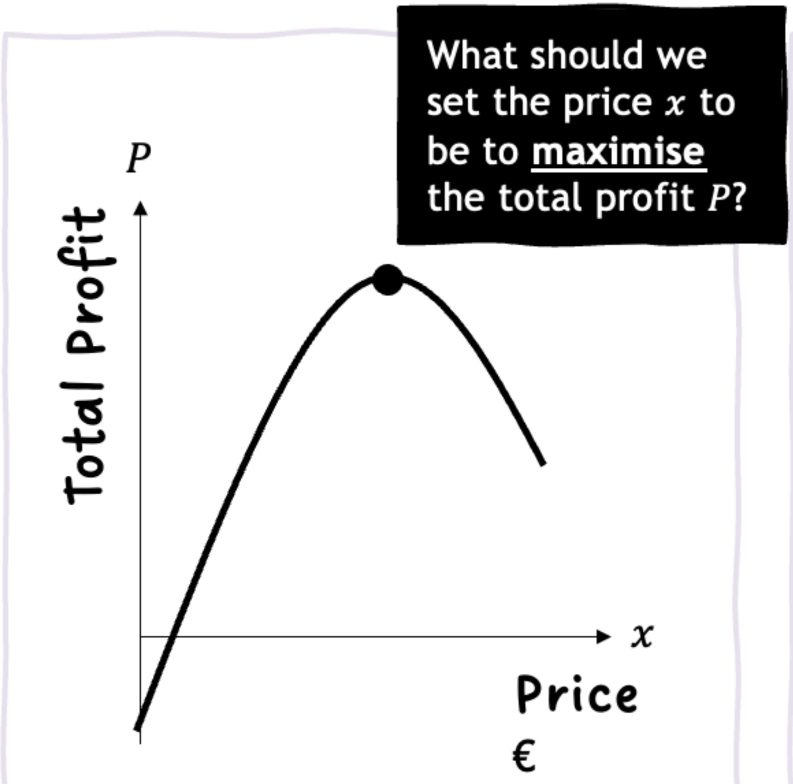
Fill in the Gaps

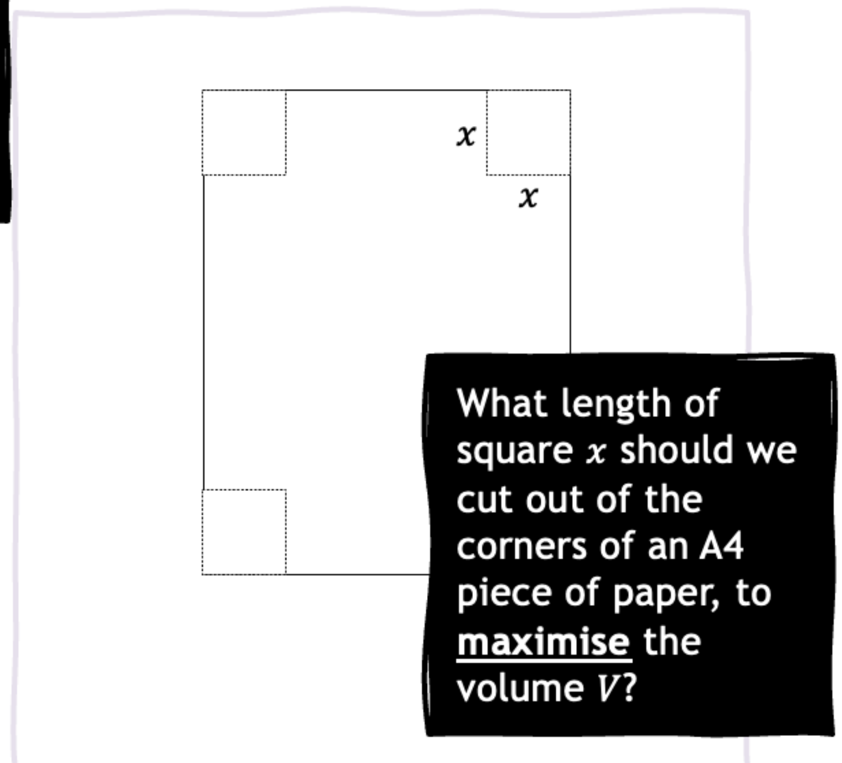
Function	Derivative	Derivative at $x = 2$	2^{nd} Derivative	Function at $x = 1$	Stationary Point(s)
$y = x^2 - 2x + 10$					
	$6x^2$			2	
	$5x$			$-\frac{3}{2}$	
		8	$6x - 2$	-10	
	$x^2 - 3x + 2$			1	
			-2	7	$x = 3$
	3			5	

Optimisation

What is Optimisation?

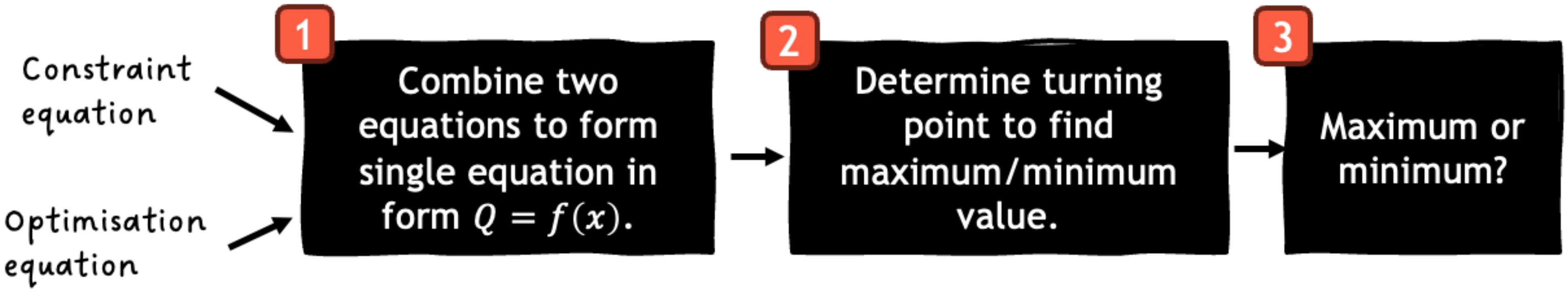
Identifying minimum or maximum points becomes practically useful when trying to **minimise/maximise some real-world output,** given **we can control some input**.





These are examples of **optimisation problems**.

Typical Format of Optimisation Questions

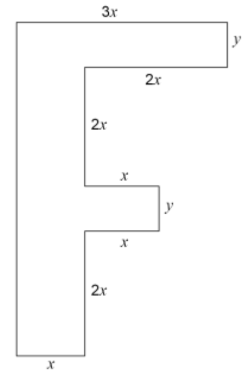


Worked Example	Your Turn
$U = 81y + \frac{49}{y}, y > 0$ Use calculus to show that V has a minimum value and work out the minimum value of V	$V = 49x + \frac{81}{x}, x > 0$ Use calculus to show that V has a minimum value and work out the minimum value of V

Worked Example

An F shape is made from rectangles.
All lengths are in centimetres.
The perimeter of the shape is $128\ cm$
The area of the shape is $A\ cm^2$

a) Find an expression for y
b) Hence find an expression for A.
c) Use calculus to derive an expression for the rate of change of A as x varies.
d) Hence work out the maximum area



Your Turn

An F shape is made from rectangles.

All lengths are in centimetres.

The perimeter of the shape is $64\ cm$

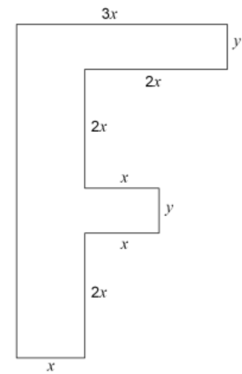
The area of the shape is $A\ cm^2$

a) Find an expression for y

b) Hence find an expression for A.

c) Use calculus to derive an expression for the rate of change of A as x varies.

d) Hence work out the maximum area



Worked Example

A cuboid is to be made with volume $81\ cm^3$

The cuboid has a rectangular cross-section where the length of the rectangle is equal to twice its width, x cm

The volume of the cuboid is $81\ cm^3$

a) Show that the total length, L, of the twelve edges of the cuboid is given by $L = 12x + \frac{162}{x^2}$

b) Given that x can vary, use differentiation to find the maximum or minimum value of L

c) Justify that the value of L you have found is a minimum

Your Turn

A cuboid is to be made from $54m^2$ of sheet metal.
The cuboid has a horizontal base and no top.
The height of the cuboid is x metres.
Two of the opposite vertical faces are squares.

a) Show that the volume, V m^3, of the tank is given by $V = 18x - \frac{2}{3}x^3$

b) Given that x can vary, use differentiation to find the maximum or minimum value of V

c) Justify that the value of V you have found is a maximum

Graph Sketching

Worked Example	Your Turn
Sketch the following graph, labelling all intercept(s) and any turning point(s): $y = 2x^3 - 3x^2 - 11x + 6$	Sketch the following graph, labelling all intercept(s) and any turning point(s): $y = 2x^3 - 3x^2 - 11x + 6$

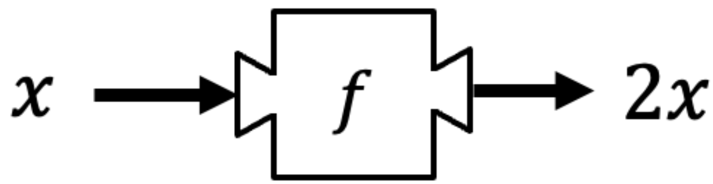
Dr Frost 517i

Extra Notes

2 Domain and Range

Functions

Functions are a mapping from an input to an output.



This simple function f takes an input x and outputs double the value.

Input → x, Output → $2x$

$$f(x) = 2x$$

Input → x, Output → $2x$

$$f : x \to 2x$$

Here, x refers to the input and $f(x)$ refers to the output.

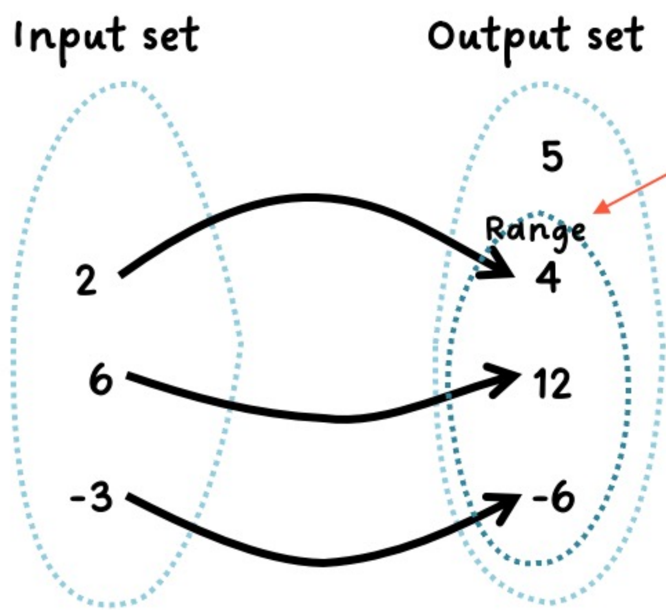
You might also see this notation. It means "the value of f is $x \to 2x$, i.e. a mapping from x to $2x$"

Domain and Range

It is often helpful about what set of values might be input into the function, and the resulting set of possible outputs.

The set of possible inputs of a mapping is known as the **domain**.

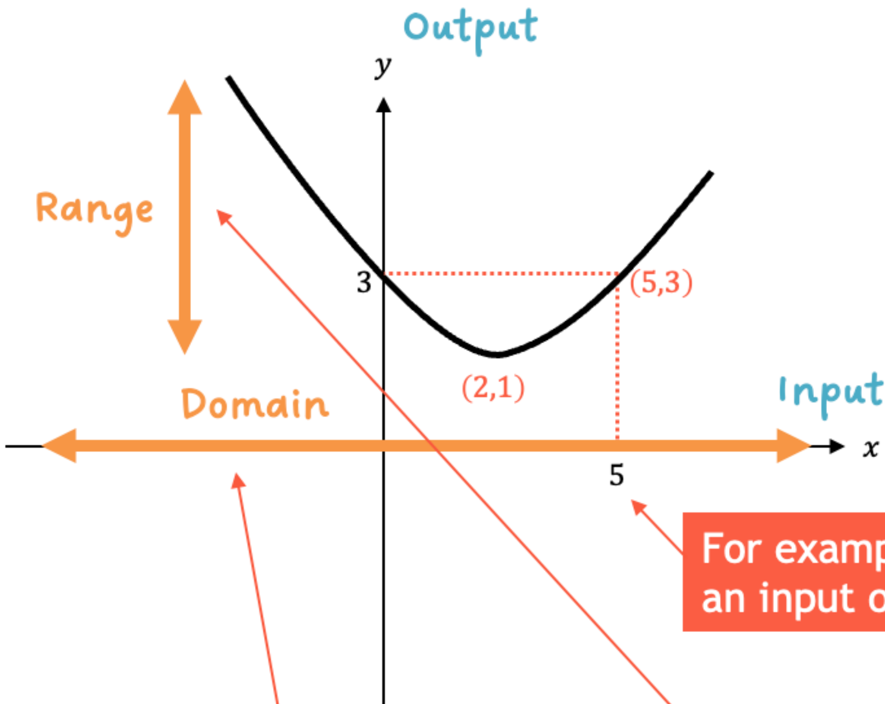
The set of **possible** outputs of a mapping is known as the **range**.*



* We can define the output set to be whatever we like, e.g. $\mathbb{R}$, known as the **codomain**. The **range** is a subset of the codomain and is only the values that can **actually be mapped to**, so wouldn't include the 5.
In A Level/IB, **you can assume the output set is the range** and there will not be any 'stray' values in the output set.

Worked Example	Your Turn
$f(x) = 3x^2 - 2$ The domain of $f(x)$ is $\{1, 2, 3, 4\}$. What is the range?	$g(x) = 2x^3 + 1$ The domain of $g(x)$ is $\{1, 2, 3, 4\}$. What is the range?

Representing Functions as Graphs



A function can also be **represented as a graph**, using the x-axis as the inputs and the y-axis as the outputs.

For example, if (5,3) is a point on the graph, an input of 5 corresponds to an output of 3.

The domain is the possible inputs, so is the horizontal span of the graph.

The range is the possible outputs, so is the vertical span of the graph.

Domain: $\boldsymbol{x} \in \mathbb{R}$

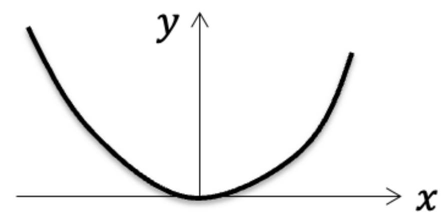
This means "the input x can be any real number".

Range: $\boldsymbol{f(x)} \in \mathbb{R}, \boldsymbol{f(x)} \geq \mathbf{1}$

Since $f(x)$ refers to the output of the function, **your range should be in terms of $f(x)$**

Method

$f(x) = x^2$ Sketch:



Note: By 'suitable', I mean the *largest* possible set of values that could be input into the function.

Suitable Domain: for all x

We can use any real number as the input!
In 'proper' maths we'd use $x \in \mathbb{R}$ to mean "x can be any element in the set of real numbers", but the syllabus is looking for "for all x".

Range: $f(x) \geq 0$

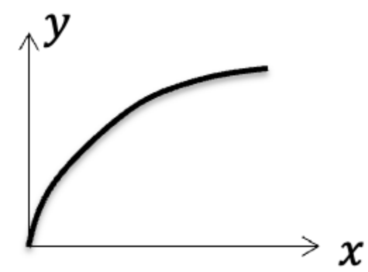
Look at the y values on the graph.
The output has to be positive, since it's been squared.

Tip: Note that the domain is in terms of x and the range in terms of $f(x)$.

Method

$$f(x) = \sqrt{x}$$

Sketch:



Suitable Domain:

$$x \geq 0$$

Presuming the output has to be a real number, we can't input negative numbers into our function.

Range:

$$f(x) \geq 0$$

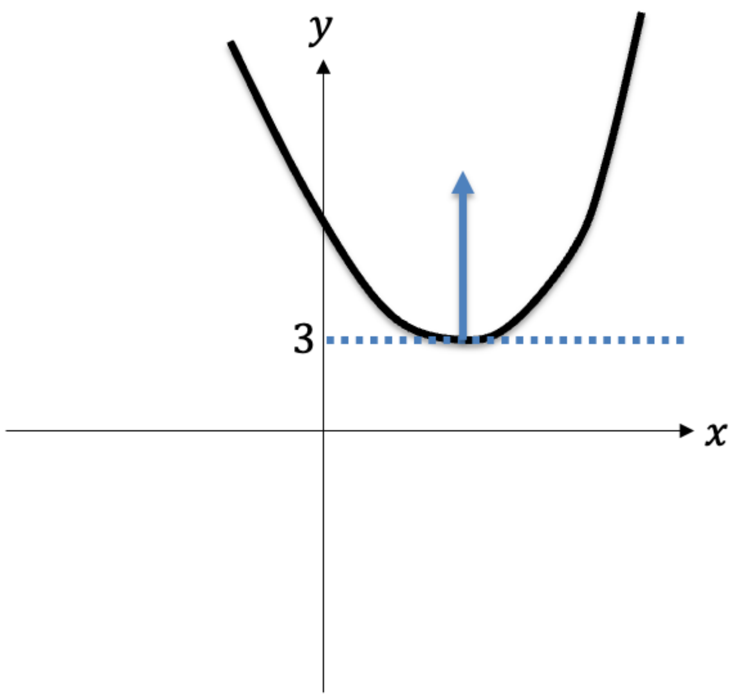
The output, again, can only be positive.

Worked Example	Your Turn
a) Work out a suitable domain and the range of $f(x) = x^2$ b) Work out a suitable domain and the range of $f(x) = \sqrt{x}$	a) Work out a suitable domain and the range of $f(x) = x^3$ b) Work out a suitable domain and the range of $f(x) = \sqrt[3]{x}$

Worked Example	Your Turn
$f(x) = \frac{x-3}{5x-6}$ State the value of x that cannot be in the domain of $f(x)$	$f(x) = \frac{3-x}{6x-5}$ State the value of x that cannot be in the domain of $f(x)$

Range of Quadratics

A common exam question is to determine the range of a quadratic.



The sketch shows the function $y = f(x)$ where $f(x) = x^2 - 4x + 7$.
Determine the range of $f(x)$.

We need the minimum point, since from the graph we can see that y (i.e. $f(x)$) can be anything greater than this.

$$\boldsymbol{f(x) = (x-2)^2 + 3}$$

The minimum point is $(2, 3)$ thus the range is:

$$\boldsymbol{f(x) \geq 3}$$

(note the $\geq$ rather than $>$)

An alternative way of thinking about it, once you've completed the square, is that anything squared is at least 0. So if $(x-2)^3$ is at least 0, then clearly $(x-2)^2 + 3$ is at least 3.

Worked Example	Your Turn
$g(x) = x^2 - 6x + 5, x \in \mathbb{R}$ Determine the range of $g(x)$	$f(x) = x^2 - 4x + 7, x \in \mathbb{R}$ Determine the range of $f(x)$

Worked Example	Your Turn
$g(x) = 3x^2 - 2x + 4, x \in \mathbb{R}$ Determine the range of $g(x)$	$f(x) = 2x^2 + 7x - 7, x \in \mathbb{R}$ Determine the range of $f(x)$

Worked Example	Your Turn
$g(x) = 14 + 2x - x^2, x \in \mathbb{R}$ Determine the range of $g(x)$	$f(x) = 21 + 4x - x^2, x \in \mathbb{R}$ Determine the range of $f(x)$

Dr Frost 564h

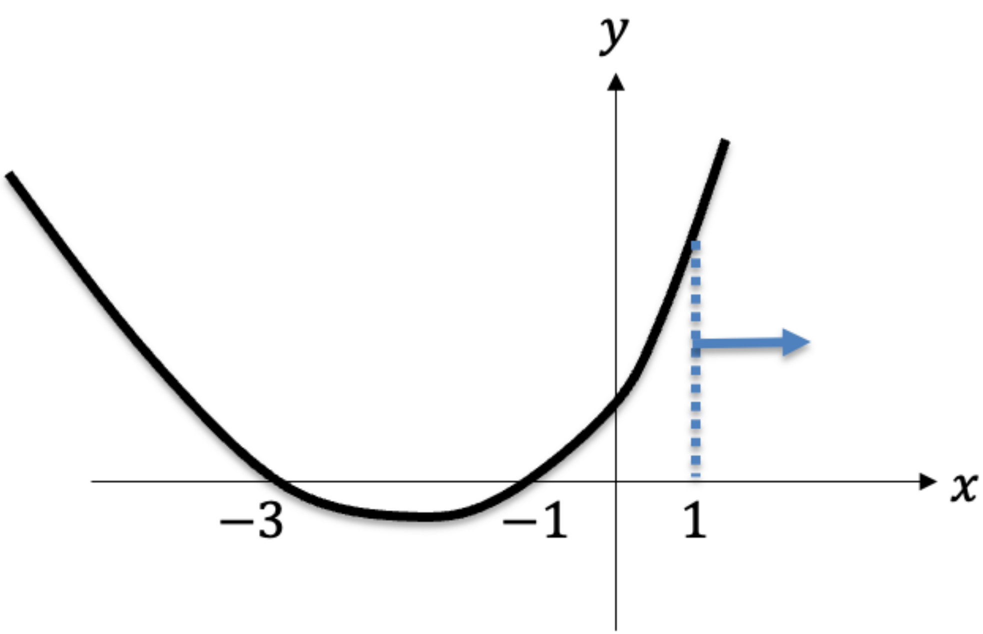
Worked Example	Your Turn
$f(x)$ is a function with domain all values of x $f(x) = \sqrt{x^2 + 12x - a}$ where a is a constant Work out the possible values of a	$f(x)$ is a function with domain all values of x $f(x) = \sqrt{x^2 + 6x - a}$ where a is a constant Work out the possible values of a

Range for Restricted Domains

Some questions are a bit jammy by restricting the domain. Look out for this, because it affects the domain!

$$f(x) = x^2 + 4x + 3, \qquad x \geq 1$$

Determine the range of $f(x)$.



Notice how the domain is $x \geq 1$.

$$\boldsymbol{f(x) = (x+1)(x+3)}$$

When $x = 1, y = 1^2 + 4 + 3 = 8$

Sketching the graph, we see that when $x = 1$, the function is increasing.

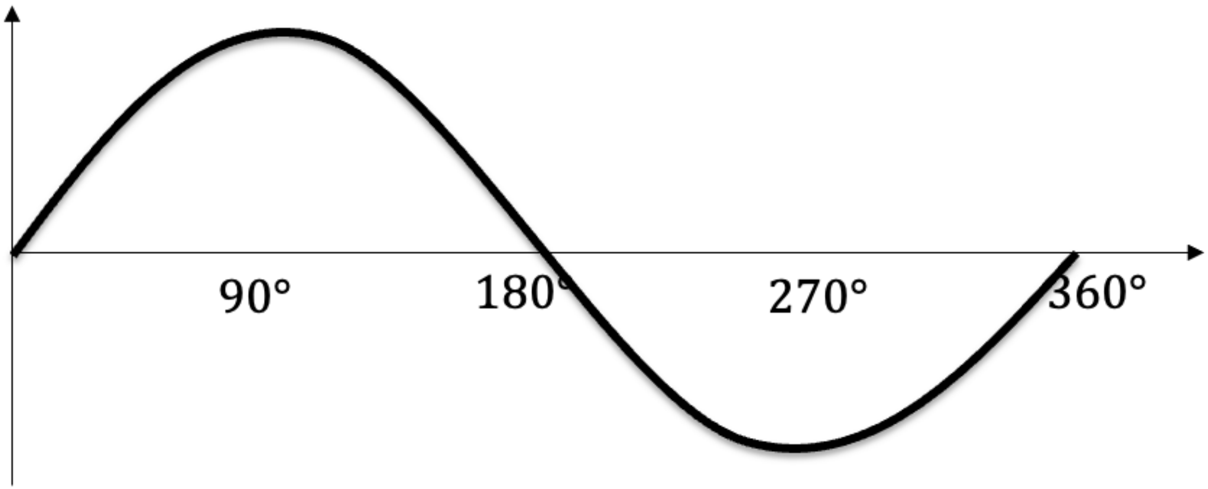
Therefore when $x \geq 1$,

$$\boldsymbol{f(x) \geq 8}$$

Worked Example	Your Turn
$g(x) = x^2 + 6x + 5, x \geq 2$ Determine the range of $g(x)$	$f(x) = x^2 + 4x + 3, x \geq 1$ Determine the range of $f(x)$

Worked Example	Your Turn
$f(x) = 23 - 5x$ with domain $-3 < x \leq 1$ Work out the range of $f(x)$	$g(x) = 32 - 3x$ with domain $-5 \leq x < 2$ Work out the range of $g(x)$

Range of Trigonometric Functions



Suppose we restricted the domain in different ways.
Determine the range in each case (or vice versa). Ignore angles below 0 or above 360.

Domain	Range
For all x (i.e. unrestricted)	$-1 \leq f(x) \leq 1$
$180 \leq x \leq 360$	$-1 \leq f(x) \leq 0$
$0 \leq x \leq 180$	$0 \leq f(x) \leq 1$

Worked Example	Your Turn
Determine the range of: $g(x) = \sin x, 180 \leq x < 360°$	Determine the range of: $f(x) = \cos x, 180 < x \leq 360°$

Worked Example	Your Turn
A function f is defined by $f(x) = \sin x,\ 0 \leq x \leq k°$ Given that the range is $0 \leq f(x) \leq 1$, determine the minimum value of k	A function f is defined by $f(x) = \cos x,\ k \leq x \leq 360°$ Given that the range is $0 \leq f(x) \leq 1$, determine the minimum value of k

Worked Example	Your Turn
$g(x) = \frac{8x-2}{x-1}, x \geq 7$ Work out the range of g	$f(x) = \frac{5x+3}{x-4}, x \geq 5$ Work out the range of f

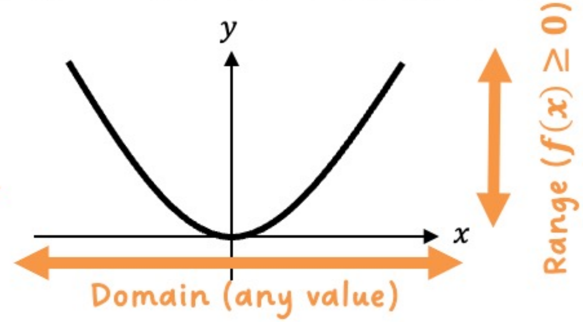
Dr Frost 564j

Domain and Range of Common Functions

By 'suitable', we mean the biggest possible set of inputs.

Function	Suitable Domain	Range
Linear $f(x) = ax + b$	$x \in \mathbb{R}$	$f(x) \in \mathbb{R}$
Quadratic $f(x) = x^2$	$x \in \mathbb{R}$	$f(x) \geq 0$
Cubic $f(x) = x^3$	$x \in \mathbb{R}$	$f(x) \in \mathbb{R}$
Exponential $f(x) = a^x$	$x \in \mathbb{R}$	$f(x) > 0$
Reciprocal $f(x) = \frac{1}{x}$	$x \neq 0$	$f(x) \neq 0$
Sinusoidal $f(x) = \sin x$	$x \in \mathbb{R}$	$-1 \leq f(x) \leq 1$
Square Root $f(x) = \sqrt{x}$	$x \geq 0$	$f(x) \geq 0$

Do a quick sketch of the graph if you can't visualise it.



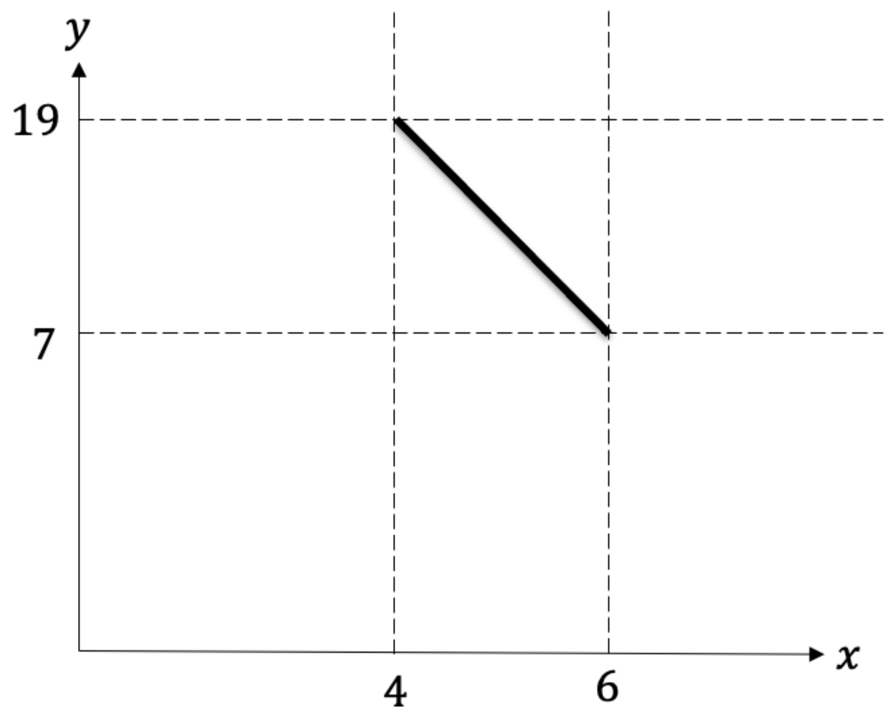
Exponential graphs have an asymptote $y = 0$. The graph tends towards a y value of 0 but never reaches it.

This is because of the input is 0, we get $\frac{1}{0}$, which is not well defined.

Constructing a Function from Domain/Range

Sometimes there's the additional constraint that the function is 'increasing' or 'decreasing'. We'll cover this in more depth when we do calculus, but the meaning of these words should be obvious.

$f(x)$ is a decreasing function with domain $4 \leq x \leq 6$ and range $7 \leq f(x) \leq 19$.



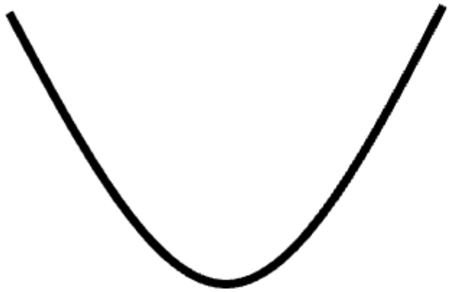
$$\boldsymbol{m = -\frac{12}{2} = -6}$$
$$\boldsymbol{y - 7 = -6(x - 6)}$$
$$\boldsymbol{y = -6x + 43}$$
$$\boldsymbol{f(x) = 43 - 6x}$$

Worked Example	Your Turn
$g(x)$ is an increasing function with domain $1 \leq x \leq 5$ and range $3 \leq g(x) \leq 11$. Construct a suitable function.	$f(x)$ is a decreasing function with domain $4 \leq x \leq 6$ and range $7 \leq f(x) \leq 19$. Construct a suitable function.

Extra Notes

3 Piecewise Functions

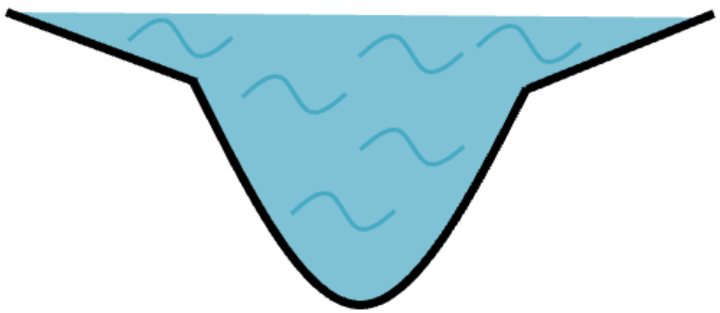
Introduction



$f(x) = x^2$

$f(x) = \frac{1}{2}x$

Previously, we've been able to assign a single expression to a function, to get one type of graph at a time.



$f(x) = ?$

But there are more complex graph shapes, e.g. modelling the depth of a riverbed, where we'd want to join simple graph segments together.

Is there a way of defining the function **as a single entity**?

Piecewise Functions

A function f is defined as

$$f(x) = \begin{cases} 5 & \text{if } x \leq 0 \\ 5 + 4x - x^2 & \text{if } 0 < x \leq 5 \\ x - 5 & \text{if } x > 5 \end{cases}$$

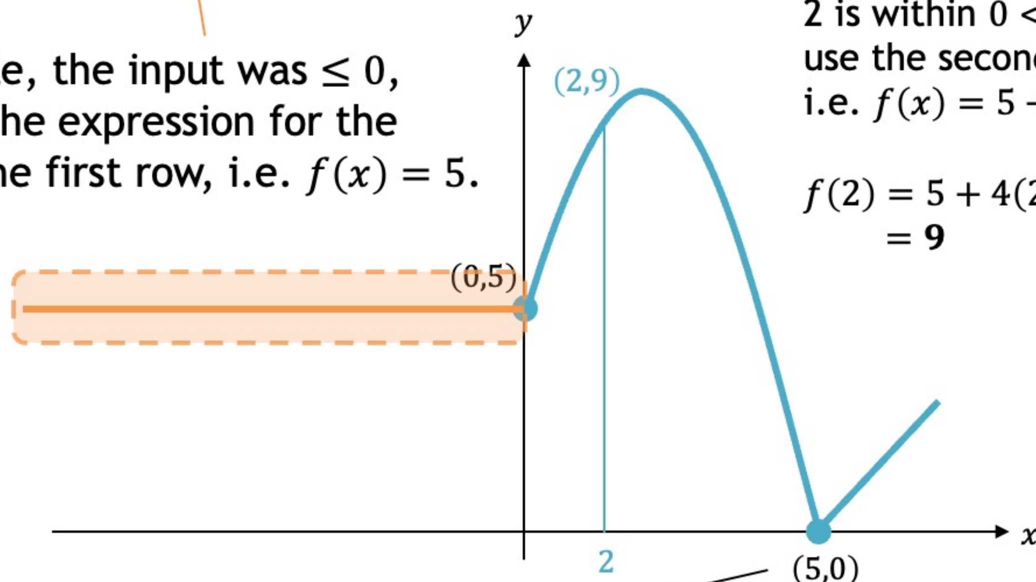
Determine $f(2)$

This is known as a **piecewise function**, where the function is defined in 'pieces' for different parts of the domain. Each row represents a different piece.

We want $f(2)$.
2 is within $0 < x \leq 5$, so we use the second row, i.e. $f(x) = 5 + 4x - x^2$.

$$f(2) = 5 + 4(2) - 2^2$$
$$= \mathbf{9}$$

If, for example, the input was ≤ 0, then we use the expression for the function on the first row, i.e. $f(x) = 5$.



Typically, the boundaries between the parts of the graph (e.g. $x = 0$ and $x = 5$) match (i.e. the lines 'join up'), so $f(5)$ would be the same whether we use $f(x) = 5 + 4x - x^2$ or $f(x) = x - 5$. But the graph will not necessarily be 'continuous' and may make sudden jumps in y value. In our example, (5,0) would be part of $f(x) = 5 + 4x - x^2$ as 5 satisfies $0 < x \leq 5$ but not $x > 5$

Worked Example	Your Turn
$f(x) = \begin{cases} x^2 + 4, & -8 \leq x \leq 0 \\ 3x + 4, & 0 < x \leq 7 \end{cases}$ Work out the value of $f(-3)$	$f(x) = \begin{cases} x^2 + 1, & 0 \leq x \leq 3 \\ 2x + 4, & 3 < x \leq 8 \end{cases}$ Work out the value of $f(7)$

Dr Frost 487a

Sketching Piecewise Linear Functions

The function f is defined as

$$f(x) = \begin{cases} 3 & \text{if } x \leq 1 \\ 4 - x & \text{if } 1 < x \leq 5 \\ 2 & \text{if } x > 5 \end{cases}$$

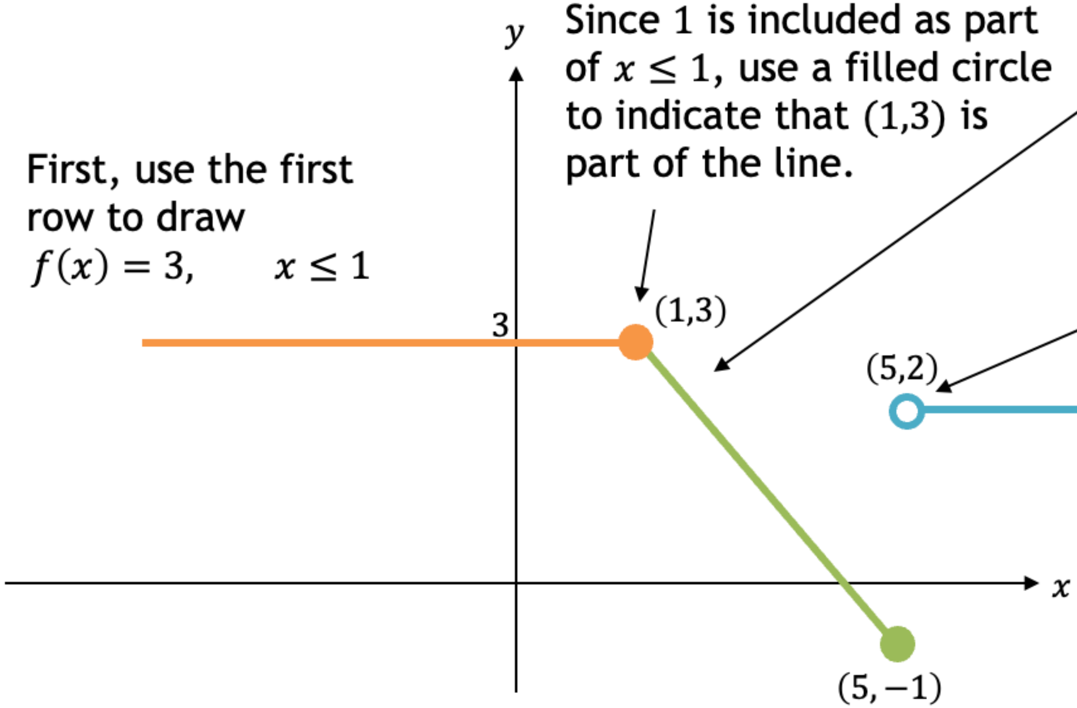
Sketch the graph with equation $y = f(x)$

First, use the first row to draw
$f(x) = 3, \quad x \leq 1$

Since 1 is included as part of $x \leq 1$, use a filled circle to indicate that $(1,3)$ is part of the line.

Next, sketch
$f(x) = 4 - x, \quad 1 < x \leq 5$
Use the endpoints of the interval:
$f(1) = 3 \rightarrow (1,3)$
$f(5) = -1 \rightarrow (5,-1)$

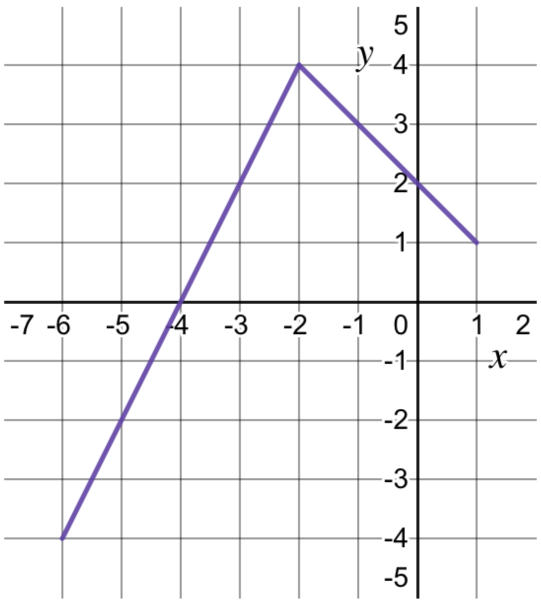
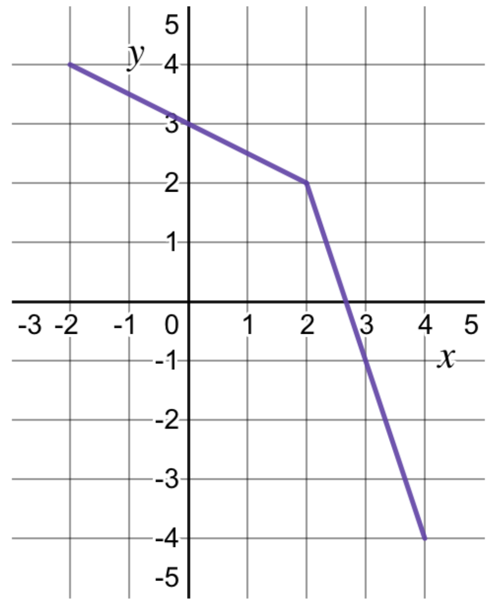
Finally sketch
$f(x) = 2, \quad x > 5$
Because $x = 5$ is not part of this interval, and because there is a **discontinuity** (the y value suddenly changes), use a hollow circle to indicate that this line segment does **not** include this point.



Worked Example	Your Turn
$f(x) = \begin{cases} (x-2)^2 + 1, & 0 \le x < 3 \\ \frac{1}{4}x + \frac{5}{4}, & 3 \le x \le 7 \end{cases}$ Sketch the graph of $y = f(x)$	$f(x) = \begin{cases} (x-1)^2 + 2, & 0 \le x < 2 \\ \frac{1}{3}x + \frac{7}{3}, & 2 \le x \le 5 \end{cases}$ Sketch the graph of $y = f(x)$

Worked Example	Your Turn
$f(x)=\begin{cases}3, & 0\le x<1\\ x^2+2, & 1\le x<2\\ 8-x, & 2\le x<3\end{cases}$ Sketch the graph of $y=f(x)$	$f(x)=\begin{cases}x^2, & 0\le x<1\\ 1, & 1\le x<2\\ 3-x, & 2\le x<3\end{cases}$ Sketch the graph of $y=f(x)$

Dr Frost 487b

Worked Example	Your Turn
The function $f(x)$ is defined as $$f(x) = \begin{cases} p, & -6 \leq x \leq -2 \\ q, & -2 < x \leq 1 \end{cases}$$ where p and q are unknown expressions. The graph of $y = f(x)$ is shown below.  Find the expressions represented by p and q.	The function $f(x)$ is defined as $$f(x) = \begin{cases} p, & -2 \leq x \leq 2 \\ q, & 2 < x \leq 4 \end{cases}$$ where p and q are unknown expressions. The graph of $y = f(x)$ is shown below.  Find the expressions represented by p and q.

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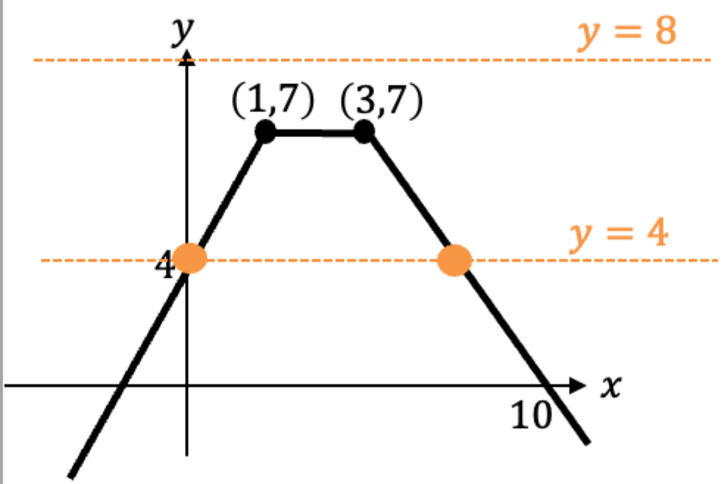
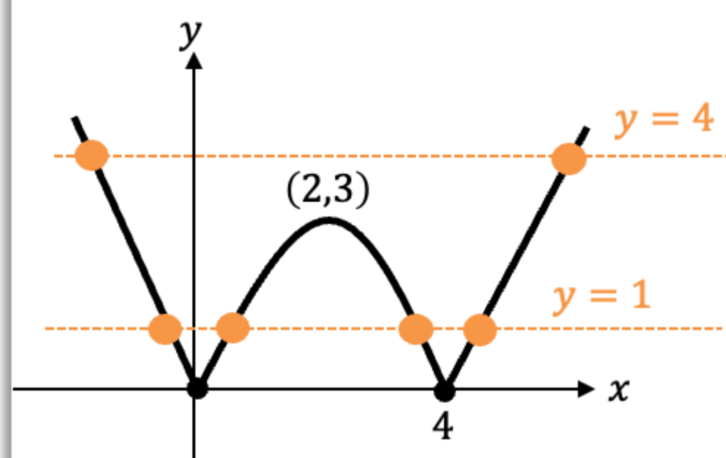
Worked Example	Your Turn
$f(x) = \begin{cases} (x-a)^2 + b, & 0 \leq x < 3 \\ cx + d, & 3 \leq x \leq 7 \end{cases}$ The graph of $y = f(x)$ passes through the points $(0, 5)$, $(2, 1)$, $(3, 2)$ and $(7, 3)$ Find the values of a, b, c and d	$f(x) = \begin{cases} (x-a)^2 + b, & 0 \leq x < 2 \\ cx + d, & 2 \leq x \leq 5 \end{cases}$ The graph of $y = f(x)$ passes through the points $(0, 3)$, $(1, 2)$, $(2, 3)$ and $(5, 3)$ Find the values of a, b, c and d

Fluency Practice

By observation, state how many solutions there will be to each of the following equations.



a $f(x) = 4$
b $f(x) = 8$



a $f(x) = 1$
b $f(x) = 4$

a ?
b ?

a ?
b ?

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Worked Example	Your Turn
The function $f(x)$ is defined as $f(x) = \begin{cases} x^2 + 7, & -2 \leq x \leq 2 \\ 15 - 2x, & 2 < x \leq 7 \end{cases}$ Solve $f(x) = 8$	The function $f(x)$ is defined as $f(x) = \begin{cases} -2x - 10, & -4 \leq x \leq -2 \\ -x^2 - 2, & -2 < x \leq 2 \end{cases}$ Solve $f(x) = -3$

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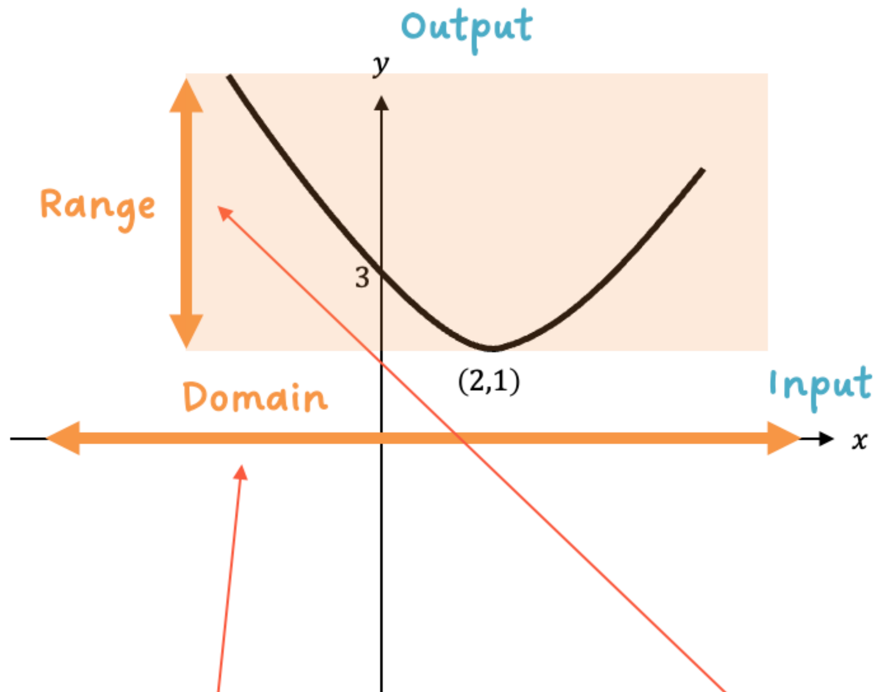
Range of a Function



The domain is the **possible inputs**, so is the horizontal span of the graph.

The range is the possible outputs, so is the vertical span of the graph.

Domain: **all real x**

The input x can be any number. We could also write this as $x \in \mathbb{R}$ which means "x is a member of the set of real numbers"

Range: $f(x) \geq 1$

Since $f(x)$ refers to the output of the function, your range should be in terms of $f(x)$

Worked Example	Your Turn
The function $f(x)$ is defined for all x: $f(x)=\begin{cases} 9, & x<-3 \\ x^2, & -3\leq x\leq 3 \\ 15-2x, & x>3 \end{cases}$ Determine the range of $f(x)$	The function $f(x)$ is defined for all x: $f(x)=\begin{cases} 4, & x<-2 \\ x^2, & -2\leq x\leq 2 \\ 12-4x, & x>2 \end{cases}$ Determine the range of $f(x)$

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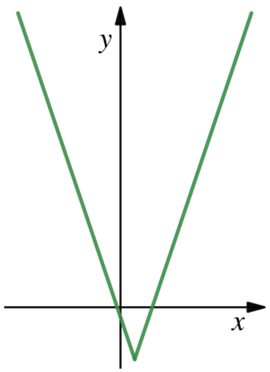
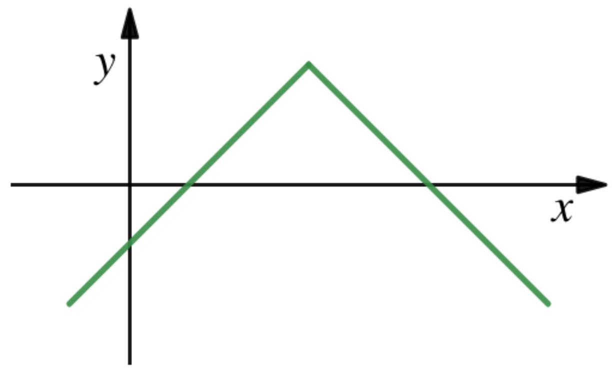
Worked Example	Your Turn
The function $f(x)$ is defined as $f(x) = \begin{cases} p, & -7 \leq x \leq 1 \\ q, & 1 < x \leq 9 \end{cases}$ where p and q are unknown expressions. The graph of $y = f(x)$ is shown below.  The graph is symmetrical about $x = 1$. The range of $f(x)$ is $-6 \leq f(x) \leq 34$. Find the expressions represented by p and q.	The function $f(x)$ is defined as $f(x) = \begin{cases} p, & -1 \leq x \leq 3 \\ q, & 3 < x \leq 7 \end{cases}$ where p and q are unknown expressions. The graph of $y = f(x)$ is shown below.  The graph is symmetrical about $x = 3$. The range of $f(x)$ is $-2 \leq f(x) \leq 2$. Find the expressions represented by p and q.

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Worked Example	**Your Turn**
The function $f(x)$ is defined as $f(x) = \begin{cases} 2, & 0 \leq x \leq 5 \\ 2x - 8, & 5 < x \leq 7 \\ 27 - 3x, & 7 < x \leq 9 \end{cases}$ Find the area enclosed by the graph of $y = f(x)$, the y-axis and the x-axis.	The function $f(x)$ is defined as $f(x) = \begin{cases} 2, & 0 \leq x \leq 5 \\ x - 3, & 5 < x \leq 8 \\ 45 - 5x, & 8 < x \leq 9 \end{cases}$ Find the area enclosed by the graph of $y = f(x)$, the y-axis and the x-axis.

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Extra Notes