KING EDWARD VI HANDSWORTH GRAMMAR SCHOOL FOR BOYS

KING EDWARD VI ACADEMY TRUST BIRMINGHAM

Year 12
Pure Mathematics
P2 5 Radians
Booklet

HGS Maths

Dr Frost Course

Name: ________________________________

Class: ________________________________

Contents

Prior knowledge check

Prior knowledge check

1 Write down the exact values of the following trigonometric ratios.

a $\cos 120°$ b $\sin 225°$ c $\tan(-300°)$

d $\sin(-480°)$ ← Year 1, Chapter 10

2 Simplify each of the following expressions.

a $(\tan\theta\cos\theta)^2 + \cos^2\theta$

b $1 - \dfrac{1}{\cos^2\theta}$ c $\sqrt{1 - \dfrac{\sin\theta\cos\theta}{\tan\theta}}$

← Year 1, Chapter 10

3 Show that

a $(\sin 2\theta + \cos 2\theta)^2 \equiv 1 + 2\sin 2\theta\cos 2\theta$

b $\dfrac{2}{\sin\theta} - 2\sin\theta \equiv \dfrac{2\cos^2\theta}{\sin\theta}$

← Year 1, Chapter 10

4 Solve the following equations for θ in the interval $0 \leqslant \theta \leqslant 360°$, giving your answers to 3 significant figures where they are not exact.

a $4\cos\theta + 2 = 3$ b $2\sin 2\theta = 1$

c $6\tan^2\theta + 10\tan\theta - 4 = \tan\theta$

d $10 + 5\cos\theta = 12\sin^2\theta$

← Year 1, Chapter 10

5.1 Radian Measure

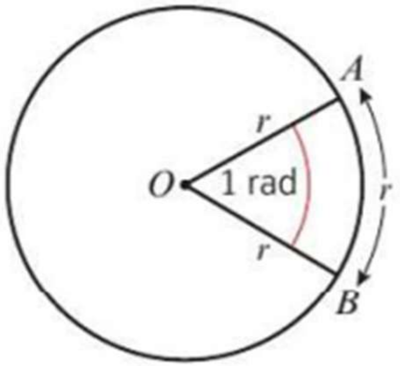
A degree is an *arbitrary* measure of an angle which comes from the idea that once there was believed to be 360 days in a year.

A radian on the other hand is not arbitrary as it represents that angle such that a single unit ensures a sectors arc length is equal to the radii.

Therefore, as circumference is $2\pi r$ it follows that

$$360° = 2\pi \text{ radians}$$

If the arc AB has length r, then $\angle AOB$ is 1 radian.



- **2π radians = 360°**
- **π radians = 180°**
- **1 radian = $\frac{180°}{\pi}$**

Notes

Worked Example

Convert to radians:

a) $360°$

b) $45°$

c) $120°$

d) $315°$

e) $72°$

Worked Example

Convert to degrees:

a) 5π

b) $\frac{\pi}{3}$

c) $\frac{5\pi}{6}$

d) $\frac{9\pi}{4}$

e) $\frac{4\pi}{5}$

Worked Example

Convert the following angles into degrees.

a) $\frac{7\pi}{8}$ rad

b) $\frac{4\pi}{15}$ rad

Worked Example

Convert the following angles into radians. Leave your answers in terms of π.

a) $150°$

b) $110°$

Worked Example

Find:

a) $\sin(0.3 \text{ rad})$

b) $\cos(\pi \text{ rad})$

c) $\mathrm{t}an(2 \text{ rad})$

Give your answers to 2 decimal places where appropriate.

Worked Example

Sketch the graph for $0 \leq x \leq 2\pi$ of $y = \sin x$

Worked Example

Sketch the graph for $0 \leq x \leq 2\pi$ of $y = \cos\ x$

Worked Example

Sketch the graph for $0 \leq x \leq 2\pi$ of $y = \tan\ x$

Worked Example

Sketch the graph for $0 \leq x \leq 2\pi$ of $y = \cos\left(x + \frac{\pi}{2}\right)$

Worked Example
Sketch the graph for $0 \leq x \leq 2\pi$ of $y = \cos(x + \pi)$

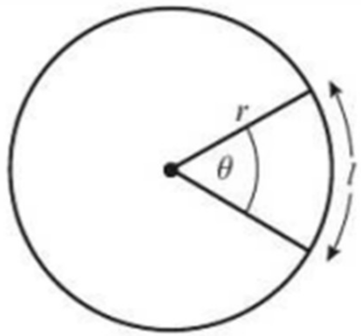
Worked Example

Sketch the graph for $0 \leq x \leq 2\pi$ of $y = \sin(2x)$

5.2 Arc Length

Using radians greatly simplifies the formula for **arc length**.

- **To find the arc length l of a sector of a circle use the formula $l = r\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.**



Notes

Worked Example

Find the length of the arc of a circle of radius 5.2 cm, given that the arc subtends an angle of 0.8 radians at the centre of the circle.

Worked Example

An arc AB of a circle with radius 7 cm and centre O has a length of 2.45 cm. Find the angle $\angle AOB$ subtended by the arc at the centre of the circle.

Worked Example

An arc AB of a circle, with centre O and radius r cm, subtends an angle of θ radians at O. The perimeter of the sector AOB is P cm. Express r in terms of P and θ.

Worked Example

The border of a garden pond consists of a straight edge AB of length 2.4 m, and a curved part C, also connecting A and B. The curve part is an arc of a circle, centre O, radius 2 m.
Find the length of C

Worked Example

A triangle ABC is such that $AB = 8\ cm$, $AC = 11\ cm$ and $\angle BAC = 0.7$ radians.
The arc BD, where D lies on AC, is an arc of a circle with centre A and radius 8 cm.
A region R, is bounded by the straight lines BC and CD and the arc BD.
Find the perimeter of R

Worked Example

A sector of a circle of radius 15 cm contains an angle of θ radians. Given that the perimeter of the sector is 42 cm, find the value of θ

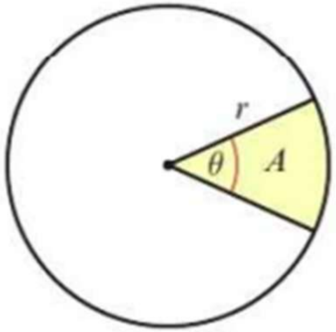
Worked Example

The perimeter of a sector OAB is four times the length of the arc AB. Find the size of angle AOB

5.3 Areas of Sectors and Segments

Using radians also greatly simplifies the formula for the area of a **sector**.

- **To find the area A of a sector of a circle use the formula $A = \frac{1}{2}r^2\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.**



Notes

Worked Example

Find the area of the sector of a circle of radius 2.44cm, given that the sector subtends an angle of 1.4 radians at the centre of the circle.

Worked Example

A circle, centre O, radius 5.2 cm has a minor sector OAB where the arc AB subtends an angle of 0.8 radians at the centre of the circle.
Find the area of the sector.

Worked Example

A circle, centre O, radius 5.2 cm has a minor sector OAB where the arc AB subtends an angle of 0.8 radians at the centre of the circle.
A segment is enclosed by a chord AB and the arc AB.
Find the area of the segment.

Worked Example

The area of the minor sector AOB is 28.9 cm^2. Given that $\angle AOB = 0.8$ radians and O is the centre of the circle, calculate the length of the radius

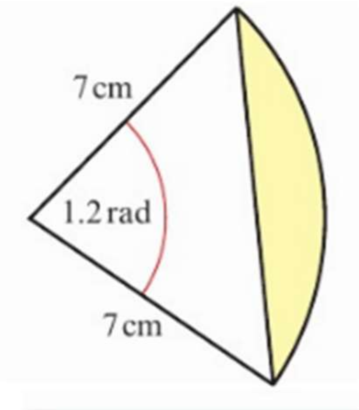
Worked Example

A sector of a circle of radius 55 m and perimeter 176 m.

Calculate the area of the sector

Worked Example

The diagram shows a sector of a circle. Find the area of the shaded segment.



Worked Example

OAB is a sector of a circle, centre O, radius $4m$.
The chord AB is 5m long.
Find the area of the segment.

Worked Example

AB is the diameter of a semicircle, centre O, radius r cm.
C is a point on the semicircle.
<BOC $= \theta$ radians.
Given that the area of ΔAOC is three times the segment enclosed by CB, show that $3\theta - 4\sin\theta = 0$

Worked Example

OAB is a sector of a circle, centre O, radius 9 cm and angle 0.7 radians.
C lies outside the sector.
AC is a straight line, perpendicular to OA.
OBC is a straight line.
Find the area of the region bounded by the arc AB and the lines AC and BC

Worked Example

OPQ is a sector of a circle, centre O, radius 10 cm where <POQ $= 0.3$ radians.
The point R is on OQ such that the ratio OR:RQ is $1:3$
A region is bounded by the arc PQ, QR and a line RP.

a) Find the perimeter of the region
b) Find the area of the region

5.4 Solving Trigonometric Equations

Notes

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:

$$\sin\theta = \frac{1}{2}$$

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:

$$\sin\theta + 1 = \frac{1}{2}$$

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:
$3 \sin \theta + 1 = 0.4$

Worked Example

Find the solutions of these equations in the interval $0 \leq \theta < 2\pi$:

a) $\sin\theta = 0.3$

b) $4\cos\theta = 2$

c) $5\tan\theta + 3 = 1$

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:

$\sin(\theta - \frac{\pi}{4}) = \frac{1}{2}$

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:

$\sin 3\theta = \dfrac{\sqrt{3}}{2}$

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:

$\sin^2\theta = \frac{1}{4}$

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:

$2\sin^2\theta - 5\sin\theta - 3 = 0$

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:
$5\sin^2 \theta - 2 \sin \theta = 0$

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:
$5\cos\theta\, sin\theta + 2\sin\theta = 0$

Worked Example

Solve in the interval $0 \leq \theta < 2\pi$:
$2\tan x = 3\sin x$

Worked Example

Find all the solutions, in the interval
$0 \leq x < 2\pi$, of the equation
$2\cos^2 x + 1 = 5\sin x$,
giving each solution in terms of π.

Your Turn

Solve the equation $17\cos\theta + 3\sin^2\theta = 13$ in the interval $0 \leq \theta \leq 2\pi$

5.5 Small Angle Approximations

You can use radians to find **approximations** for the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

- **When θ is small and measured in radians:**
  - $\sin\theta \approx \theta$
  - $\tan\theta \approx \theta$
  - $\cos\theta \approx 1 - \frac{\theta^2}{2}$

**these are given in formulae booklet*

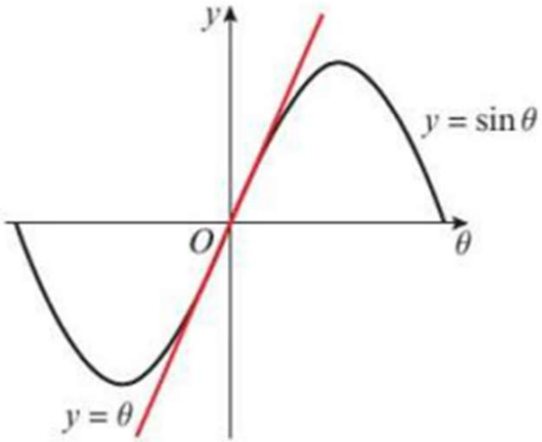
These are derived from series found in Further Maths (and the formulae booklet) where you only take the first few term(s)

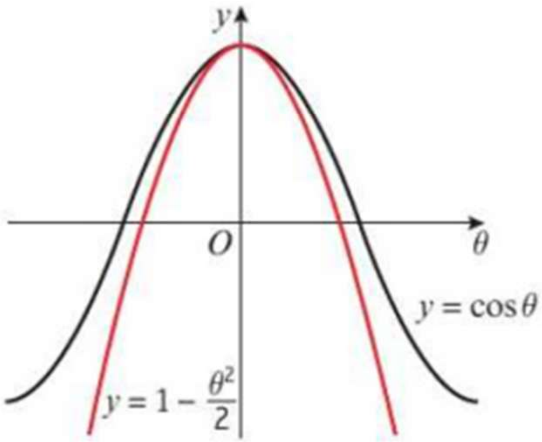
Maclaurin's and Taylor's Series

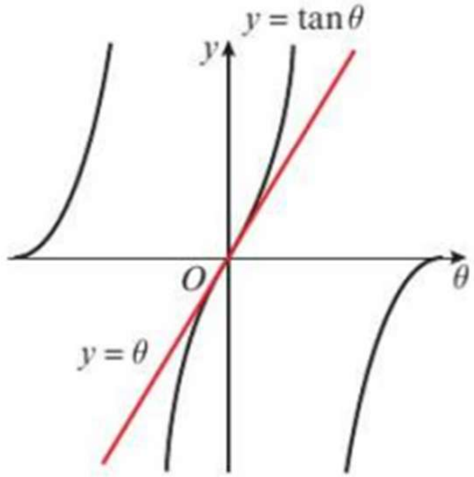
$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^r}{r!} f^{(r)}(0) + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \quad \text{for all } x$$







Notes

Worked Example

When θ is small, find the approximate value of:

a) $\frac{\sin 2\theta + \tan}{2\theta}$

b) $\frac{\cos 4\theta - 1}{\theta \sin}$

c) $\sin 5\theta + \tan 2\theta - \cos 2\theta$

Worked Example

Find the percentage error when calculating the value of $\cos(0.246\ rad)$ using the small-angle approximations.

Worked Example

When θ is small, find the approximate value of:

$$\frac{1 - 2\tan\theta - 4\cos 2\theta}{\tan 2\theta + 1}$$

Extract from Formulae book

Small angle approximations

$\sin\theta \approx \theta$

$\cos\theta \approx 1 - \frac{\theta^2}{2}$

$\tan\theta \approx \theta$

where θ is measured in radians

2.

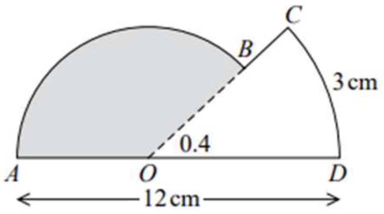


Figure 1

The shape *ABCDOA*, as shown in Figure 1, consists of a sector *COD* of a circle centre *O* joined to a sector *AOB* of a different circle, also centre *O*.

Given that arc length $CD = 3$ cm, $\angle COD = 0.4$ radians and *AOD* is a straight line of length 12 cm,

(a) find the length of *OD*,

(2)

(b) find the area of the shaded sector *AOB*.

(3)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

2(a)	Uses $s = r\theta \Rightarrow 3 = r \times 0.4$	M1	1.2
	$\Rightarrow OD = 7.5$ cm	A1	1.1b
		(2)	
(b)	Uses angle $AOB = (\pi - 0.4)$ or uses radius is $(12 - \text{"}7.5\text{"})$ cm	M1	3.1a
	Uses area of sector $= \frac{1}{2}r^2\theta = \frac{1}{2}\times(12-7.5)^2\times(\pi-0.4)$	M1	1.1b
	$= 27.8\text{cm}^2$	A1ft	1.1b
		(3)	
			(5 marks)

Summary of Key Points

1 • 2π radians = 360° • π radians = 180° • 1 radian = $\frac{180°}{\pi}$

2 • $30° = \frac{\pi}{6}$ radians • $45° = \frac{\pi}{4}$ radians • $60° = \frac{\pi}{3}$ radians

• $90° = \frac{\pi}{2}$ radians • $180° = \pi$ radians • $360° = 2\pi$ radians

3 You need to learn the exact values of the trigonometric ratios of these angles measured in radians.

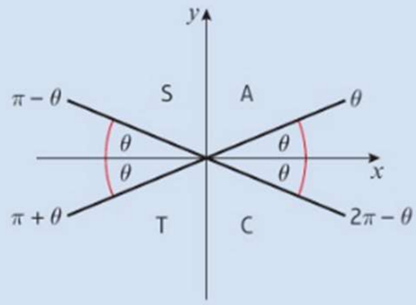
• $\sin\frac{\pi}{6} = \frac{1}{2}$ • $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ • $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

• $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ • $\cos\frac{\pi}{3} = \frac{1}{2}$ • $\tan\frac{\pi}{3} = \sqrt{3}$

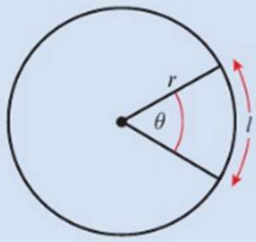
• $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ • $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ • $\tan\frac{\pi}{4} = 1$

4 You can use these rules to find sin, cos or tan of any positive or negative angle measured in radians using the corresponding acute angle made with the x-axis, θ.

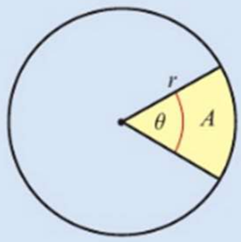
- $\sin(\pi - \theta) = \sin\theta$
- $\sin(\pi + \theta) = -\sin\theta$
- $\sin(2\pi - \theta) = -\sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\cos(\pi + \theta) = -\cos\theta$
- $\cos(2\pi - \theta) = \cos\theta$
- $\tan(\pi - \theta) = -\tan\theta$
- $\tan(\pi + \theta) = \tan\theta$
- $\tan(2\pi - \theta) = -\tan\theta$



5 To find the arc length l of a sector of a circle use the formula $l = r\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.



6 To find the area A of a sector of a circle use the formula $A = \frac{1}{2}r^2\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.



7 The area of a segment in a circle of radius r is

$$A = \frac{1}{2}r^2(\theta - \sin\theta)$$

8 When θ is small and measured in radians:

• $\sin\theta \approx \theta$ • $\tan\theta \approx \theta$ • $\cos\theta \approx 1 - \frac{\theta^2}{2}$