KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Mechanics 1
Chapter 11 – Variable Acceleration

HGS Maths

Dr Frost Course

Name: ________________________________

Class: ________________________________

Contents

Prior Knowledge Check

1 Find $\frac{dy}{dx}$ given:

a $y = 3x^2 - 5x + 6$ **b** $y = 2\sqrt{x} + \frac{6}{x^2} - 1$

← **Pure Year 1, Chapter 12**

2 Find the coordinates of the turning points on the curve with equation:

a $y = 3x^2 - 9x + 2$ **b** $y = x^3 - 6x^2 + 9x + 5$

← **Pure Year 1, Chapter 12**

3 Find $f(x)$ given:

a $f'(x) = 5x + 8$, $f(0) = 1$

b $f'(x) = 3x^2 - 2x + 5$, $f(0) = 7$

← **Pure Year 1, Chapter 13**

4 Find the area bounded by the x-axis and:

a the line $y = 2x - 1$, $x = 2$ and $x = 5$

b the curve $y = 6x - 2 - x^2$, $x = 1$ and $x = 3$

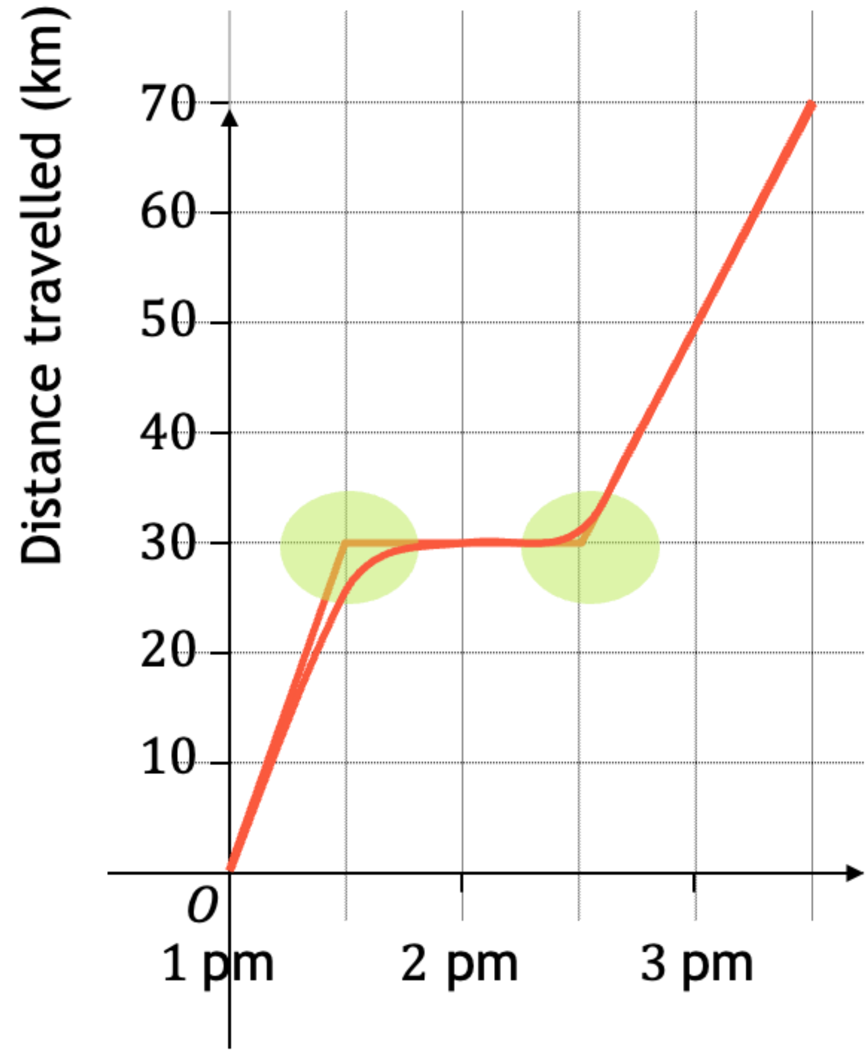
← **Pure Year 1, Chapter 13**

11.1 Functions of Time

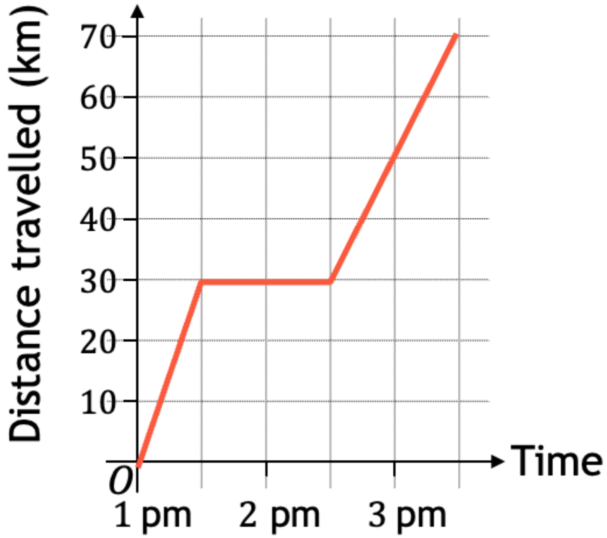
Recap – Distance-Time Graphs

Recall that when we previously drew distance-time graphs and worked with <u>constant acceleration formulae</u>, we made a modelling assumption that changes in speed (i.e. the gradient on a distance-time graph) were <u>sharp breaks</u> in the graphs.

The motion of real objects is more likely to have <u>smooth transitions</u> between different speeds.

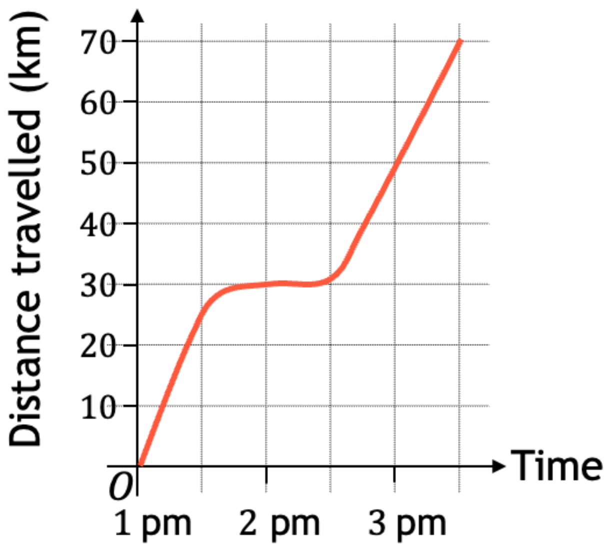


Recap – Distance-Time Graphs



The motion for the <u>simplified version</u> of the graph would look like this:

Constant speed | Instant Stop | Constant speed



Whereas this more <u>advanced model</u> would look more like this:

Deceleration | Gradual stop | Acceleration

Functions of Time

We have previously considered <u>functions of x</u>, like these:

$$y = x^3 - 8x^2 + 5$$

$$g(x) = \frac{7x + 3}{x^2 + 5x}$$

$$f(x) = \sqrt{x + 9}$$

In mechanics, we consider how the quantities <u>displacement</u>, <u>velocity</u>, and <u>acceleration</u> change with time

Therefore, we will consider these quantities as <u>functions of time</u>, t, which will look like these:

$$s = t^3 - 5t^2 + 3$$

$$v = 3t^2 - 10t$$

$$a = 6t$$

$$s(t) = 4t^2 - 6t^3$$

$$v(t) = 2t^2 - 8t$$

We read this: *'s as a function of t'*

Notes

Worked Example

A train travels along a straight track, leaving the start of the track at time $t = 0$. It then returns to the start of the track. The distance, s metres, from the start of the track at time t seconds is modelled by:

$s = 8t^2 - 5t^3, \quad 0 \leq t \leq 1.6$

Explain the restriction $0 \leq t \leq 1.6$

11.2 Using Differentiation

Recap – Differentiation Notation

Differentiate the following...

...with respect to x

$$\frac{d}{dx}(x^3) = 3x^2$$

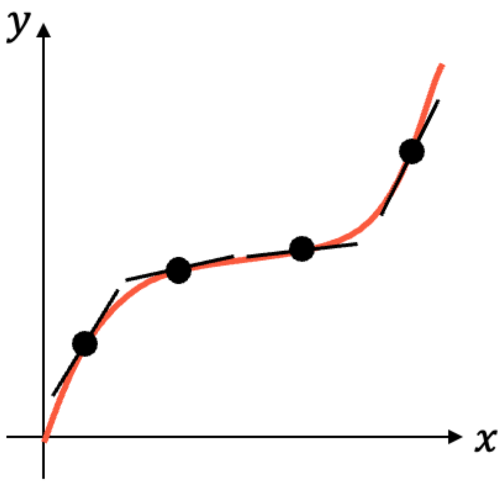
Recall from lessons on calculus that $\frac{d}{dx}(\ldots)$ means to differentiate with respect to x.

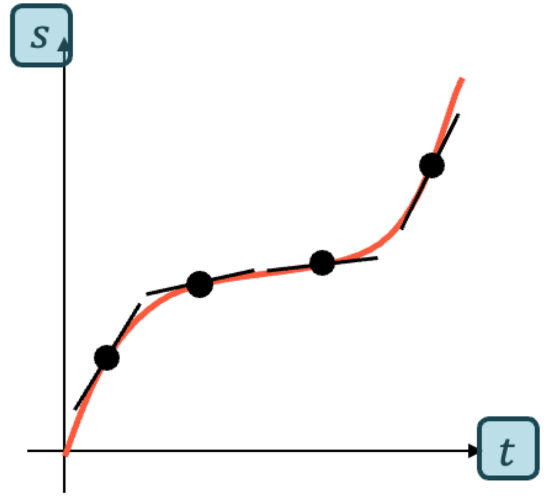
What would need to differ in order for us to differentiate this expression with respect to t ?

$$\frac{d}{dt}(t^3 + 2t^2) = 3t^2 + 4t$$

If we have a function in terms of another variable, we change what we are differentiating with respect to.

Differentiation of Kinematics Variables





If $\frac{dy}{dx}$ is the rate of change of y with respect to x , i.e. the gradient function of y

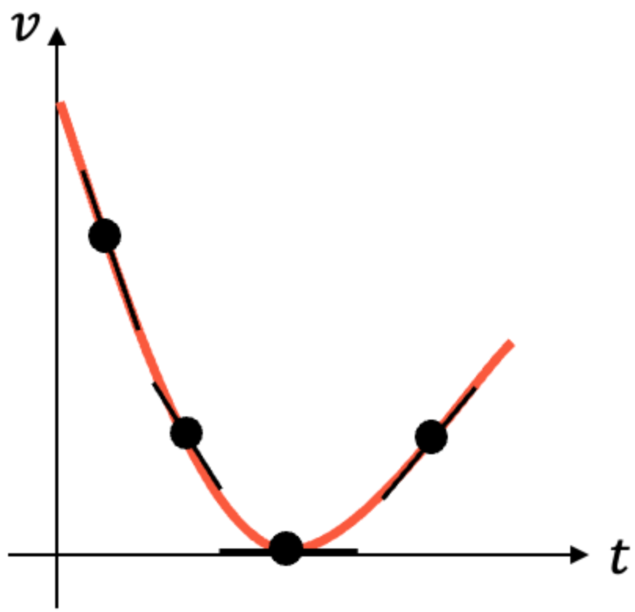
What would be true of $\frac{ds}{dt}$, if s is the displacement of an object, and t is time?

The gradient of a displacement-time graph is velocity therefore:

The gradient function for displacement with respect to time is velocity.

$$\frac{ds}{dt} = v$$

Differentiation of Kinematics Variables



What does the gradient of a velocity-time graph represent?

Acceleration

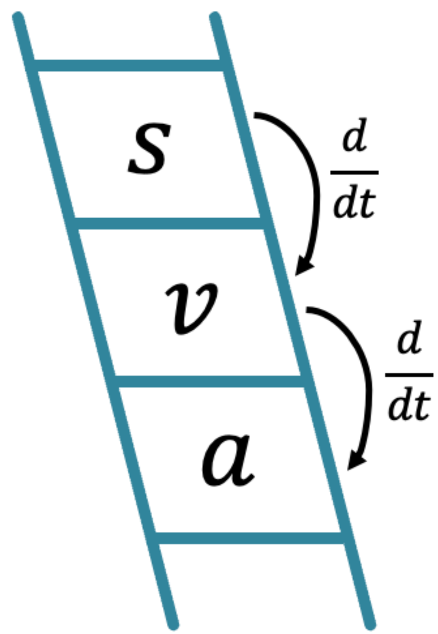
What can we therefore say about the gradient function of velocity?

The gradient function of velocity with respect to time is acceleration

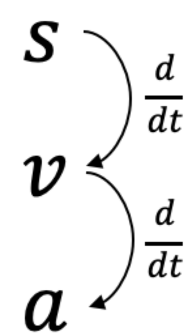
$$\frac{dv}{dt} = a$$

Differentiation of Kinematics Variables

You might visualise differentiation of the kinematics variables as descending this ladder.



EXAM TIP : It is helpful just to jot down the order like this in the corner of your page whenever you work through a variable acceleration problem.



We'll worry about integration shortly.

Notes

Your Turn

A particle P is moving on the x-axis.
At time t seconds, the displacement x metres from O is given by

$$x = \frac{1}{3}t^3 - \frac{11}{2}t^2 + 30t + 5$$

Find the distance between the two points at which the particle is at rest.

11.3 Maxima and Minima Problems

Notes

Worked Example

A child is playing with a yo-yo. The yo-yo leaves the child's hand at time $t = 0$ and travels vertically in a straight line before returning to the child's hand. The distance, s m, of the yo-yo from the child's hand after time t seconds is given by:

$s = 2.4t - 0.4t^2 - 0.4t^3, \qquad 0 \leq t \leq 2$

a) Justify the restriction $0 \leq t \leq 2$

b) Find the maximum distance of the yo-yo from the child's hand, correct to 3sf.

11.4 Using Integration

Recap – Integration Notation

Recall the way in which we write an integration:

$$\int (3x^2 + 1)\,dx = x^3 + x + c$$

Integrate… …this expression …with respect to x

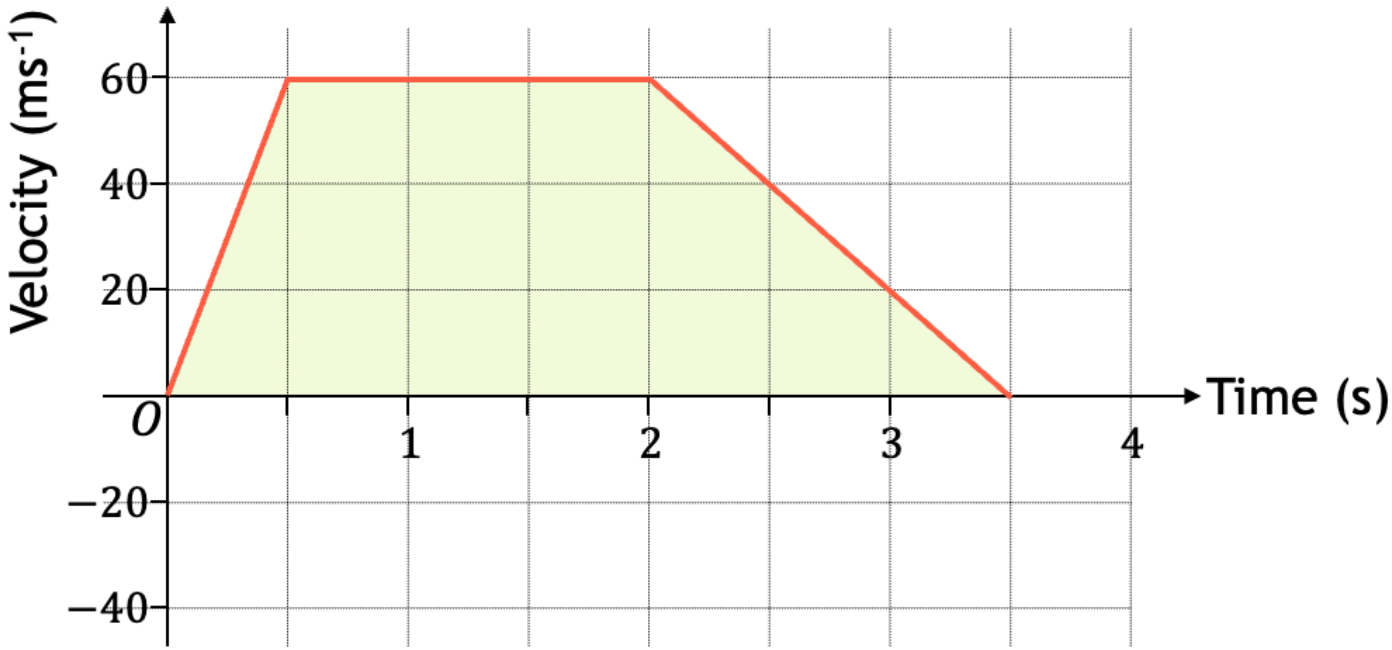
In kinematics, we will integrate expressions that are in terms of t:

$$\int (3t^2 + 1)\,dt = t^3 + t + c$$

Therefore note that we will be integrating with respect to t.

Integration of Kinematics Variables

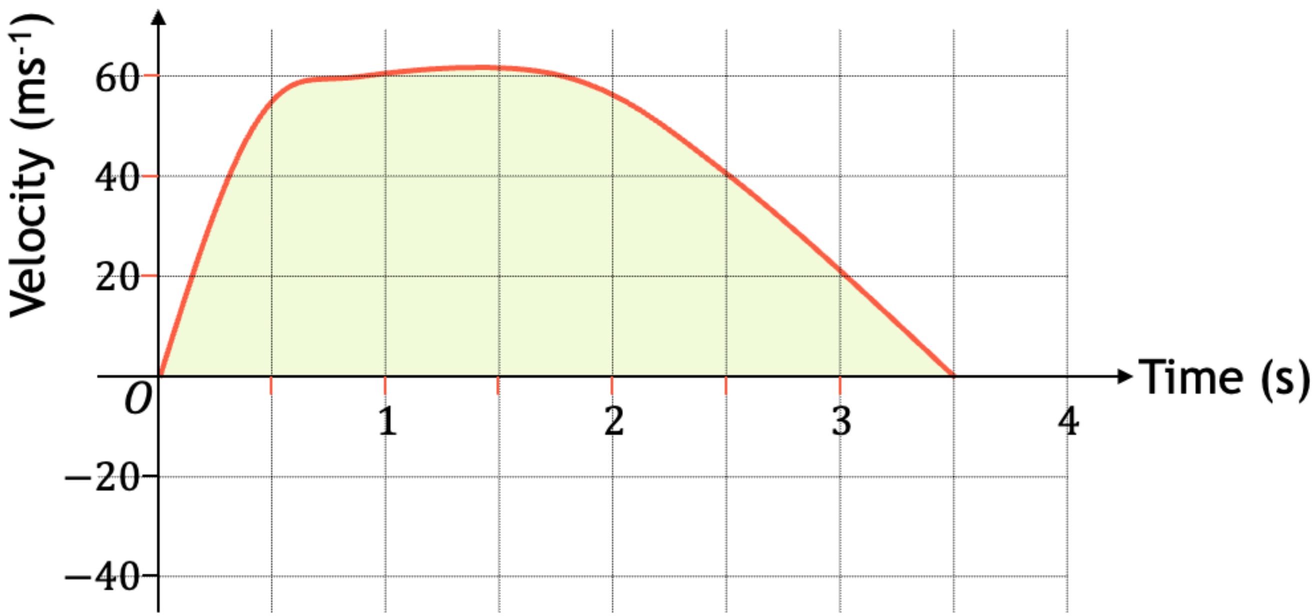
Recall : What does the area underneath a velocity-time graph represent?



The area under a velocity-time graph is equal to the displacement.

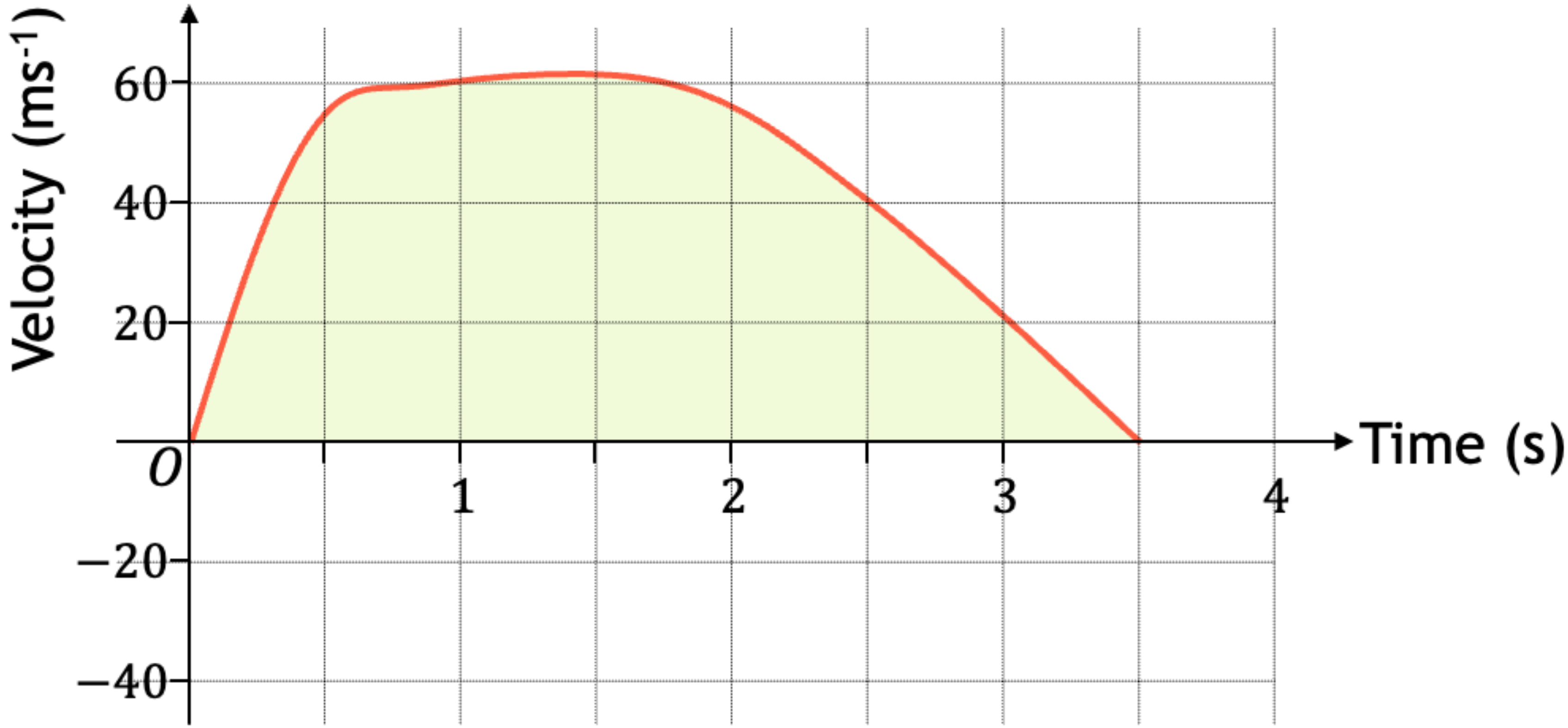
Integration of Kinematics Variables

If we had a more realistic velocity-time graph, described by a curve, how might we find the area beneath the curve?



If we knew the velocity function, we could integrate this with respect to time to find the displacement under the curve.

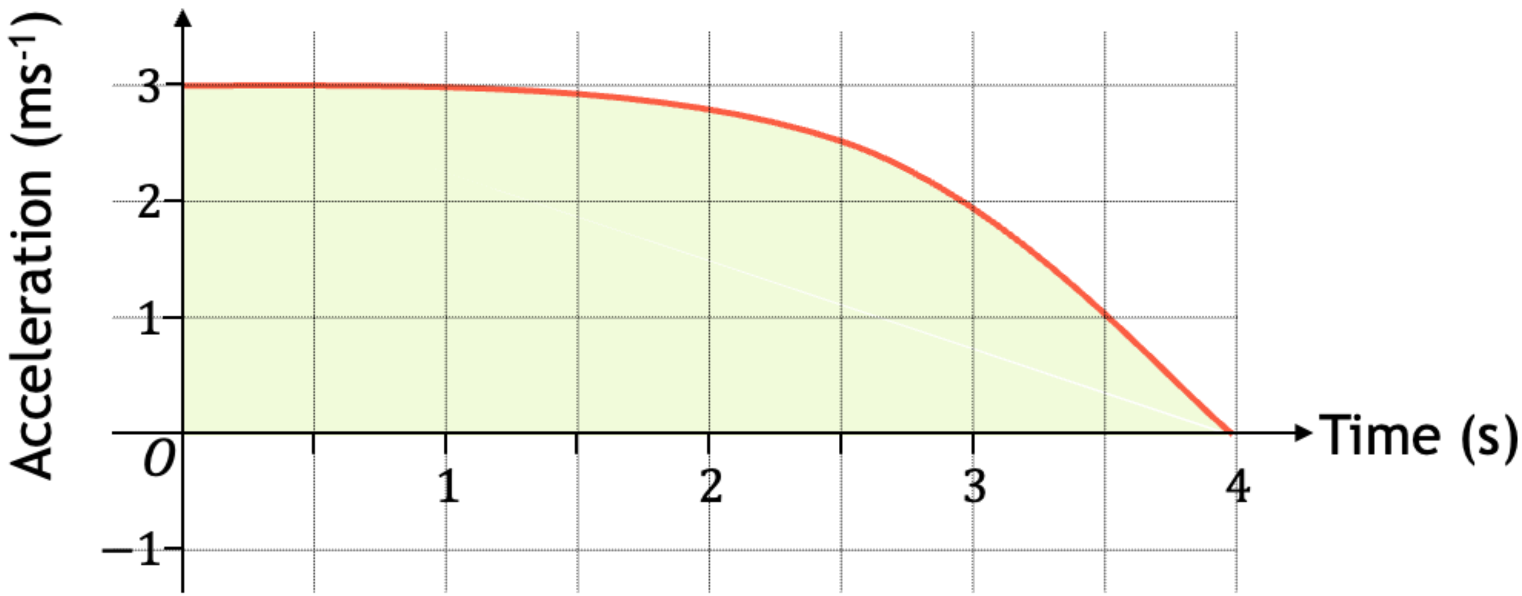
Integration of Kinematics Variables



The integral of velocity with respect to time is displacement

$$\int v \, dt = s$$

Integration of Kinematics Variables



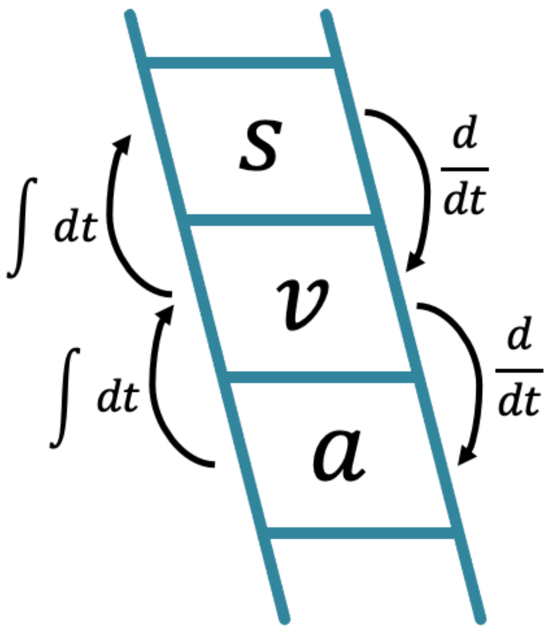
Similarly, the area under an acceleration-time graph gives us the velocity, therefore:

The integral of acceleration with respect to time is displacement

$$\int a \, dt = v$$

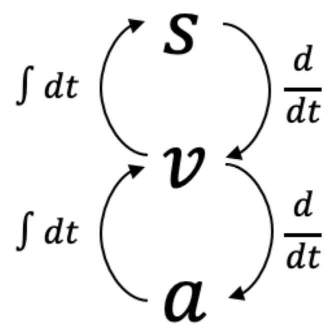
Calculus with Kinematics Variables

Referring back to the ladder visual - if we differentiate with respect to time to 'descend' the ladder...



Then we can integrate with respect to time to 'ascend' the ladder.

Noting this when you begin a variable acceleration problem can help to prompt your thinking.



Notes

Worked Example

A particle travels in a straight line.
After t seconds its velocity, v ms^{-1}, is given by $v = 7 - 6t^2$, $t \geq 0$
Find the distance travelled by the particle in the fifth second of its motion.

Worked Example

A particle travels in a straight line such that its acceleration, $a\ ms^{-2}$, at time t seconds, is given by $a = 18t + 6$
When $t = 2$ seconds, the displacement, s, is 40 metres.
When $t = 3$ seconds, the displacement is 117 metres.
Find:

a) The displacement when $t = 4$ seconds
b) The velocity when $t = 4$ seconds

Worked Example

A particle P moves along a straight line such that at time t seconds, $t \geqslant 0$, after leaving the point O on the line, the velocity, $v\,\text{m s}^{-1}$, of P is modelled as

$$v = (7 - 2t)(t + 2)$$

(a) Find the value of t at the instant when P stops accelerating.
(4)

(b) Find the distance of P from O at the instant when P changes its direction of motion.
(5)

In this question, solutions relying on calculator technology are not acceptable.

Your Turn

A particle P moves along a straight line such that at time t seconds, $t \geq 0$, after leaving the point O on the line, the velocity, v m s^{-1}, of P is modelled as

$$v = (8 - 3t)(t + 1)$$

(*a*) Find the value of t at the instant when P stops accelerating.

(4)

(*b*) Find the distance of P from O at the instant when P changes its direction of motion.

(5)

In this question, solutions relying on calculator technology are not acceptable

Worked Example

A particle P moves along a straight line.

At time t seconds, the velocity $v\,\mathrm{m\,s^{-1}}$ of P is modelled as

$$v = 10t - t^2 - k \qquad t \geqslant 0$$

where k is a constant.

(a) Find the acceleration of P at time t seconds.

(2)

The particle P is instantaneously at rest when $t = 6$

(b) Find the other value of t when P is instantaneously at rest.

(4)

(c) Find the total distance travelled by P in the interval $0 \leqslant t \leqslant 6$

(4)

Your Turn

A particle is moving on the x-axis.

At time t seconds, the velocity v m s^{-1} of the particle is modelled as

$$v = 8t - t^2 - k \qquad t \leqslant 0$$

where k is a constant.

(a) Find the acceleration of the particle at time t seconds.

(2)

The particle P is instantaneously at rest when $t = 5$

(b) Find the other value of t when the particle is instantaneously at rest.

(4)

(c) Find the total distance travelled by the particle in the interval $0 \leqslant t \leqslant 5$

(4)

Worked Example

A fixed point O lies on a straight line.

A particle P moves along the straight line.

At time t seconds, $t \geqslant 0$, the distance, s metres, of P from O is given by

$$s = \frac{1}{3}t^3 - \frac{5}{2}t^2 + 6t$$

(a) Find the acceleration of P at each of the times when P is at instantaneous rest.

(6)

(b) Find the total distance travelled by P in the interval $0 \leqslant t \leqslant 4$

(3)

Your Turn

A fixed point O lies on a straight line.

A particle P moves along the straight line.

At time t seconds, $t \geq 0$, the distance, s metres, of P from O is given by

$$s = \frac{t^3}{3} - \frac{3t^2}{2} + 2t$$

(a) Find the acceleration of P at each of the times when P is at instantaneous rest.

(6)

(b) Find the total distance travelled by P in the interval $0 \leq t \leq 4$

(3)

Worked Example

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

A fixed point O lies on a straight line.

A particle P moves along the straight line such that at time t seconds, $t \geqslant 0$, after passing through O, the velocity of P, $v\,\text{m}\,\text{s}^{-1}$, is modelled as

$$v = 15 - t^2 - 2t$$

(a) Verify that P comes to instantaneous rest when $t = 3$

(1)

(b) Find the magnitude of the acceleration of P when $t = 3$

(3)

(c) Find the total distance travelled by P in the interval $0 \leqslant t \leqslant 4$

(4)

Your Turn

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

A fixed point O lies on a straight line.

A particle P moves along the straight line such that at time t seconds, $t \geqslant 0$, after passing through O, the velocity of P, $v\,\text{m}\,\text{s}^{-1}$, is modelled as

$$v = 28 - t^2 - 3t$$

(a) Verify that P comes to instantaneous rest when $t = 4$

(1)

(b) Find the magnitude of the acceleration of P when $t = 4$

(3)

(c) Find the total distance travelled by P in the interval $0 \leqslant t \leqslant 6$

(4)

Worked Example

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

A particle is moving along a straight line.

At time t seconds, $t > 0$, the velocity of the particle is $v\,\text{m}\,\text{s}^{-1}$, where

$$v = 2t - 7\sqrt{t} + 6$$

(a) Find the acceleration of the particle when $t = 4$

(3)

When $t = 1$ the particle is at the point X.

When $t = 2$ the particle is at the point Y.

Given that the particle does not come to instantaneous rest in the interval $1 < t < 2$

(b) show that $XY = \frac{1}{3}\left(41 - 28\sqrt{2}\right)$ metres.

(4)

Your Turn

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

A particle is moving along a straight line.

At time t seconds, $t > 0$, the velocity of the particle is v m s^{-1}, where

$$v = 3t - 5\sqrt{t} + 7$$

(a) Find the acceleration of the particle when $t = 9$

(3)

When $t = 2$ the particle is at the point X.

When $t = 4$ the particle is at the point Y.

Given that the particle does not come to instantaneous rest in the interval $2 < t < 4$

(b) show that $XY = \frac{1}{3}(16 + 20\sqrt{2})$ metres.

(4)

11.5 Constant Acceleration Formulae

Notes

Worked Example

A particle moves in a straight line with constant acceleration $a\ ms^{-2}$
Given that its initial velocity is $u\ ms^{-1}$ and its initial displacement is $0\ m$, prove that:

Its velocity, $v\ ms^{-1}$, at time t s is given by $v = u + at$

Your Turn

A particle moves in a straight line with constant acceleration $a\ ms^{-2}$
Given that its initial velocity is $u\ ms^{-1}$ and its initial displacement is $0\ m$, prove that:

Its displacement, $s\ m$, at time t s is given by $s = ut + \frac{1}{2}at^2$

Summary

1 If the displacement, s, is expressed as a function of t, then the velocity, v, can be expressed as

$$v = \frac{ds}{dt}$$

2 If the velocity, v, is expressed as a function of t, then the acceleration, a, can be expressed as

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

3

Differentiate ↓		Integrate ↑
	displacement $= s = \int v \, dt$	
	$\frac{ds}{dt}$ = velocity $= v = \int a \, dt$	
	$\frac{dv}{dt} = \frac{d^2s}{dt^2}$ = acceleration $= a$	