KING EDWARD VI HANDSWORTH GRAMMAR SCHOOL FOR BOYS

KING EDWARD VI ACADEMY TRUST BIRMINGHAM

Year 12
Pure Mathematics
P1 6 Circles
Booklet

HGS Maths

Dr Frost Course

Name: ________________________________

Class: ________________________________

Contents

6.1) Midpoints and perpendicular bisectors *- recap and review*

6.2) Equation of a circle

6.3) Intersections of straight lines and circles

6.4) Use tangent and chord properties

6.5) Circles and triangles

Past Paper Practice

Summary

Prior knowledge check

Prior knowledge check

1 Write each of the following in the form $(x + p)^2 + q$:

a $x^2 + 10x + 28$ **b** $x^2 - 6x + 1$

c $x^2 - 12x$ **d** $x^2 + 7x$ ← Section 2.

2 Find the equation of the line passing through each of the following pairs of points:

a $A(0, -6)$ and $B(4, 3)$

b $P(7, -5)$ and $Q(-9, 3)$

c $R(-4, -2)$ and $T(5, 10)$ ← Section 5.

3 Use the discriminant to determine whether the following have two real solutions, one real solution or no real solutions.

a $x^2 - 7x + 14 = 0$

b $x^2 + 11x + 8 = 0$

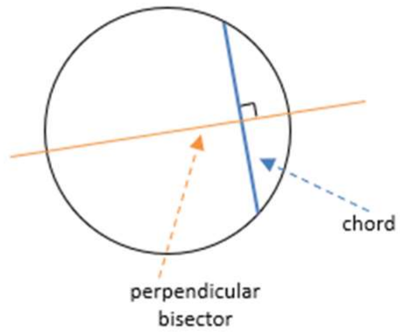
c $4x^2 + 12x + 9 = 0$ ← Section 2.

4 Find the equation of the line that passes through the point $(3, -4)$ and is perpendicula to the line with equation $6x - 5y - 1 = 0$

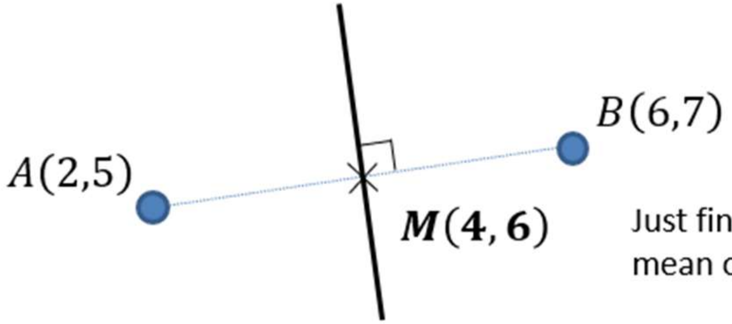
← Section 5.

6.1) Midpoints and perpendicular bisectors

Later in the chapter you will need to find the perpendicular bisector of a chord of a circle.



What two properties does a perpendicular bisector of two points A and B have?



Just find the mean of the x values and the mean of the y values.

1. It passes through the midpoint of AB.
2. It is perpendicular to AB.

Equation?

$$m_{AB} = \frac{2}{4} = \frac{1}{2}$$
$$m_{\perp} = -2$$
$$y - 6 = -2(x - 4)$$

Notes

Worked Example

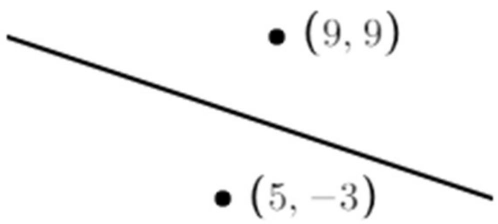
495e: Determine the perpendicular bisector of a line using $y - y_1 = m(x - x_1)$

A straight line passes through the points $A(5, -3)$ and $B(9, 9)$.

Find the equation of the perpendicular bisector of AB.

Give your answer in the form $y = mx + c$.

Simplify your answer where possible.



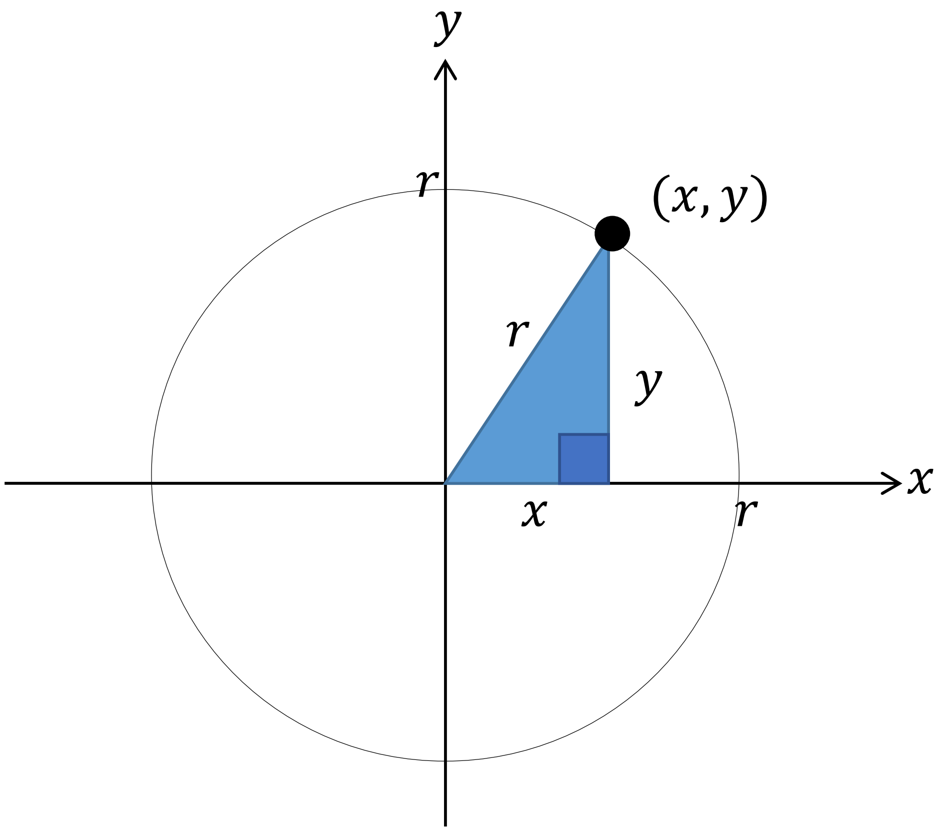
T.116: 6B Qs 1, P.46 6.1 Qs 1-5

Worked Example

A line segment AB is the diameter of a circle with centre $(4, -5)$. If A has coordinates $(2, -1)$, what are the coordinates of B?

T.116: 6B Qs 2+, P.46 6.1 Qs 6+

6.2) Equation of a circle

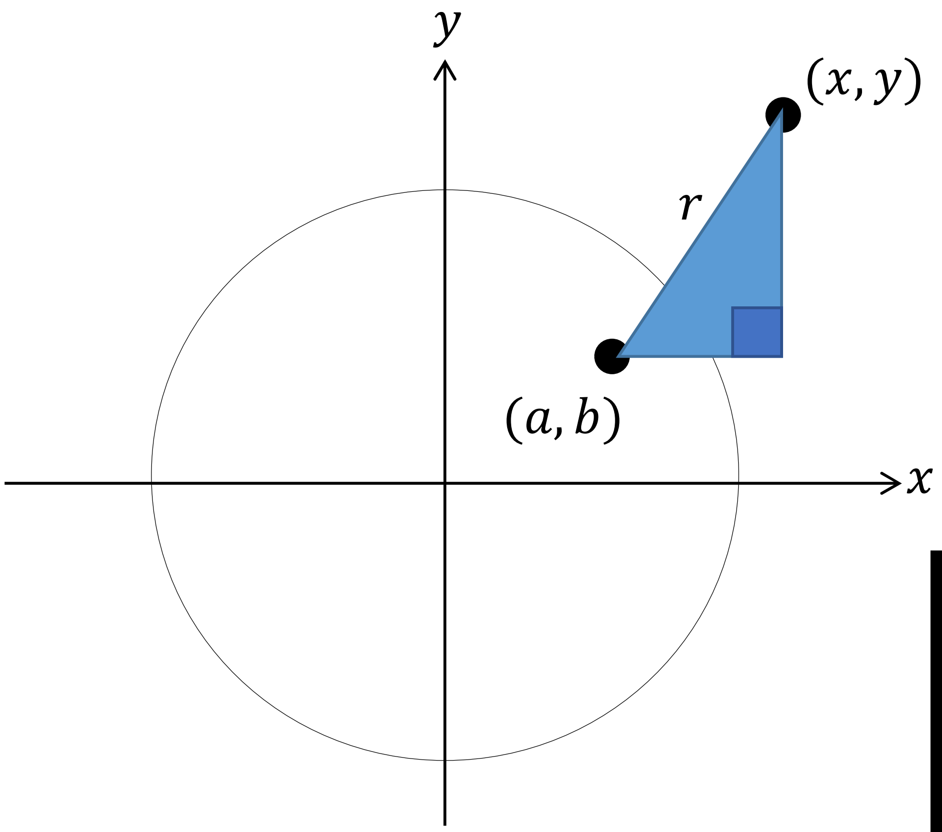


Recall that a line can be a set of points (x, y) that satisfy some equation. Suppose we have a point (x, y) on a circle centred at the origin, with radius r. What equation must (x, y) satisfy?

(Hint: draw a right-angled triangle inside your circle, with one vertex at the origin and another at the circumference)

$$x^2 + y^2 = r^2$$

Notes



Now suppose we shift the circle so it's now centred at (a, b). What's the equation now?

(Hint: What would the sides of this right-angled triangle be now?)

✏ The equation of a circle with centre (a, b) and radius r is:

$$(x - a)^2 + (y - b)^2 = r^2$$

Notes

Quickfire Questions

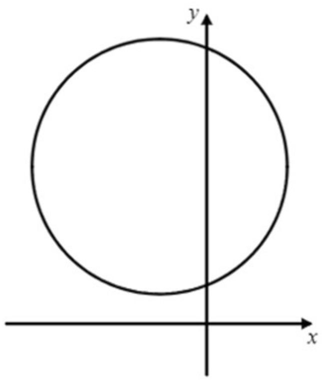
Centre	Radius	Equation
$(0,0)$	5	
$(1,2)$	6	
		$(x+3)^2+(y-5)^2=1$
		$(x+5)^2+(y-2)^2=49$
		$(x+6)^2+y^2=16$
		$(x-1)^2+(y+1)^2=3$
		$(x+2)^2+(y-3)^2=8$

Worked Example

Skill involved: 496h: Determine the intercepts of circle given its equation.

The circle C has the equation

$$(x+3)^2 + (y-10)^2 = 66$$



Find the exact y coordinates of the points where the circle C intersects the y-axis.

$y =$ ....

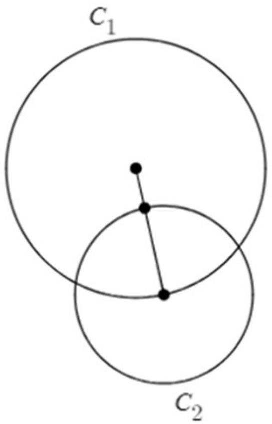
or $y =$ ....

Worked Example

Skill involved: 496i: Determine a constant in the equation of a circle given a point on the circumference.

Circle C_1 has equation $x^2 + y^2 - 4x - gy - 56 = 0$

Circle C_2 has equation $(x-4)^2 + (y+4)^2 = 40$



The centre of C_2 lies on the circumference of C_1

Find the value of g, where g is a positive constant.

Worked Example

Skill involved: 496j: Determine the equation of a circle using the endpoints of the diameter.

The line AB is the diameter of a circle where A and B have coordinates (3,12) and (19,24) respectively.

Find the equation of the circle, giving your answer in the form

$$x^2 + y^2 + ax + by + c = 0$$

Worked Example

Skill involved: 496k: Complete the square to determine the centre and radius of a circle.

A circle has equation

$$x^2 + y^2 - 2y - 120 = 0$$

Find the centre and radius of the circle

Worked Example

Skill involved: 496l: Complete the square to determine the centre and radius of a circle, where the equation involves algebraic constants.

A circle C has equation

$$x^2 + y^2 + 2kx - 4ky = -20$$

where k is a constant.

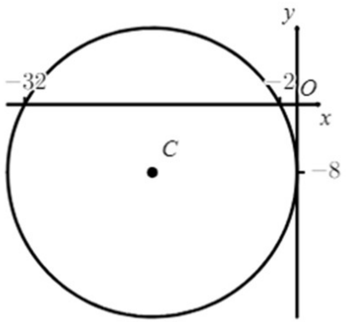
By considering the radius of C, state the range of possible values for k.

Worked Example

Skill involved: 496m: Determine the equation of a circle using simple reasoning to establish the centre or radius of the circle.

The diagram shows a circle with centre C.

The circle intersects the x-axis at $(-2,0)$ and $(0,-8)$
The circle touches the y-axis at $(-32,0)$



Work out the equation of the circle.

T.119: 6C Qs 5+, P.47 6.2 Qs 5+

11. (i) A circle C_1 has equation

$$x^2 + y^2 + 18x - 2y + 30 = 0$$

The line l is the tangent to C_1 at the point $P(-5, 7)$.

Find an equation of l in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(5)

(ii) A different circle C_2 has equation

$$x^2 + y^2 - 8x + 12y + k = 0$$

where k is a constant.

Given that C_2 lies entirely in the 4th quadrant, find the range of possible values for k.

(4)

Your Turn

11 (i) A circle C_1 has equation

$$x^2 + y^2 + 16x - 4y + 27 = 0$$

The line l is the tangent to C_1 at the point $P(-3, 6)$.

Find an equation of l in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(5)

(ii) A different circle C_2 has equation

$$x^2 + y^2 - 12x + 14y + k = 0$$

where k is a constant.

Given that C_2 lies entirely in the 4th quadrant, find the range of possible values for k.

(4)

(Total for Question 11 is 9 marks)

6.3) Intersections of straight lines and circles

Notes

Worked Example

The line with equation $y = 5x + 2$ meets the circle with equation $x^2 + kx + y^2 = 6$ at exactly one point.
Find the two possible values of k

Worked Example

The line with equation $y = 4x - 3$ does not intersect the circle with equation $x^2 + 2x + y^2 = k$.
Find the range of possible values of k.

T.122: 6D Qs all, P.48 6.3 Qs all

10. A circle C has equation

$$x^2 + y^2 + 6kx - 2ky + 7 = 0$$

where k is a constant.

(a) Find in terms of k,

(i) the coordinates of the centre of C

(ii) the radius of C

(3)

The line with equation $y = 2x - 1$ intersects C at 2 distinct points.

(b) Find the range of possible values of k.

(6)

Your Turn

10. A circle C has equation

$$x^2 + y^2 - 2kx + 4ky + 5 = 0$$

where k is a constant.

(a) Find in terms of k,

(i) the coordinates of the centre of C

(ii) the radius of C

(3)

The line with equation $y = x - 2$ intersects C at 2 distinct points.

(b) Find the range of possible values of k.

(6)

(Total for Question 10 is 9 marks)

6.4) Use tangent and chord properties

Notes

Worked Example

A circle C has equation

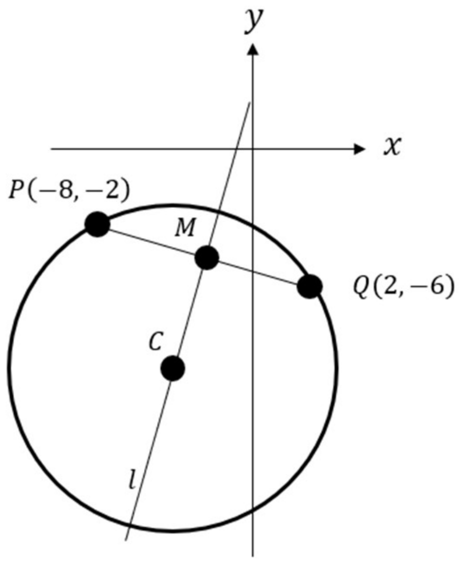
$$(x-4)^2+(y+4)^2=10$$

The line l is a tangent to the circle and has gradient -3. Find two possible equations for l, giving your answers in the form $y=mx+c$.

Worked Example

The point P has coordinates $(-8, -2)$ and the point Q has coordinates $(2, -6)$.
M is the midpoint of the line segment PQ.

a) Find an equation for l.
b) Given that the y-coordinate of C is -9:
i) show that the x-coordinate of C is -5.
ii) find an equation of the circle.



Worked Example

The line with equation $4x + y - 5 = 0$ is a tangent to the circle with equation $(x-3)^2 + (y-p)^2 = 2$.

Find the two possible values of p

Worked Example
A circle has centre $C(5,3)$, and passes through the point $P(2,6)$. Find the equation of the tangent of the circle at the point P, giving your equation in the form $ax + by + c = 0$ where a, b, c are integers..

Worked Example

A circle passes through the points $A(0,0)$ and $B(2,8)$.
The centre of the circle has x value -2. Determine the equation of the circle.

T.126: 6E Qs 6+, P.49 6.4 Qs 6+

14. A circle C with radius r

- lies only in the 1st quadrant
- touches the x-axis and touches the y-axis

The line l has equation $2x + y = 12$

(a) Show that the x coordinates of the points of intersection of l with C satisfy

$$5x^2 + (2r - 48)x + (r^2 - 24r + 144) = 0$$

(3)

Given also that l is a tangent to C,

(b) find the two possible values of r, giving your answers as fully simplified surds.

(4)

Your Turn

14. A circle C with radius r

- lies only in the 1st quadrant
- touches the x-axis and touches the y-axis

The line l has equation $y + 2x = 8$

(a) Show that the x coordinates of the points of intersection of l with C satisfy

$$5x^2 + (2r - 32)x + (r^2 - 16r + 64) = 0$$

(3)

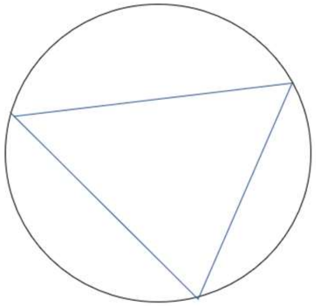
Given also that l is a tangent to C,

(b) find the two possible values of r, giving your answers as fully simplified surds.

(4)

(Total for Question 14 is 7 marks)

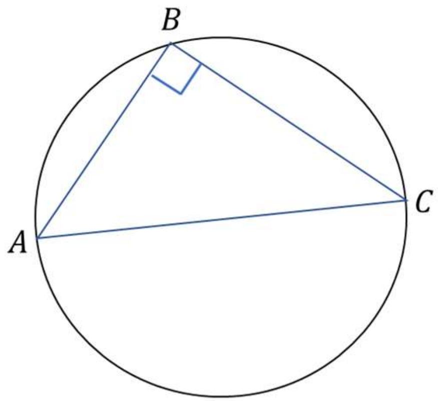
6.5) Circles and triangles



We'd say:

- The triangle **inscribes** the circle.
  (A shape inscribes another if it is inside and its boundaries touch but do not intersect the outer shape)

- The circle **circumscribes** the triangle.
- If the circumscribing shape is a circle, it is known as the **circumcircle** of the triangle.
- The centre of a circumcircle is known as the **circumcentre**.

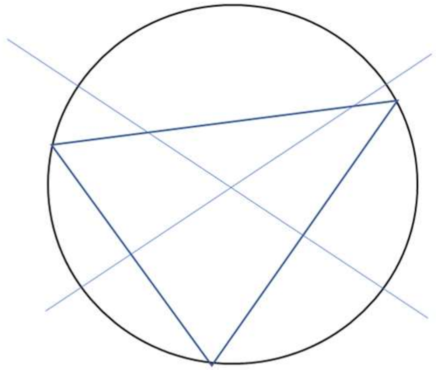


If $\angle ABC = 90°$ then:

- **AC is the diameter of the circumcircle of triangle ABC.**

Similarly if AC is the diameter of a circle:

- **$\angle ABC = 90°$ therefore AB is perpendicular to BC.**
- **$AB^2 + BC^2 = AC^2$**



Given three points/a triangle we can find the centre of the circumcircle by:

- **Finding the equation of the perpendicular bisectors of two different sides.**
- **Find the point of intersection of the two bisectors.**

Notes

Worked Example

Skill involved: 496w: Determine the equation of a circle given the coordinates of 3 points.

The coordinates $(-7,19)$, $(1,15)$ and $(-23,7)$ lie on the circle C.

Find the equation of C.

....

T.131: 6F Qs all, P.51 6.5 Qs all

Circle C_1 has equation $x^2 + y^2 = 100$

Circle C_2 has equation $(x - 15)^2 + y^2 = 40$

The circles meet at points A and B as shown in Figure 3.

(a) Show that angle $AOB = 0.635$ radians to 3 significant figures, where O is the origin.

(4)

The region shown shaded in Figure 3 is bounded by C_1 and C_2

(b) Find the perimeter of the shaded region, giving your answer to one decimal place.

(4)

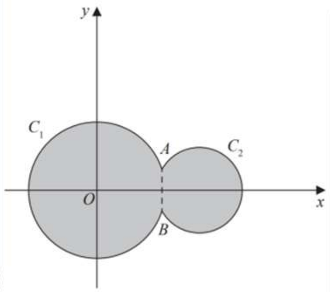


Figure 3

Your Turn

Circle C_1 has equation $x^2 + y^2 = 144$

Circle C_2 has equation $(x - 16)^2 + y^2 = 80$

The circles meet at points A and B as shown in Figure 3.

(a) Show that angle $AOB = 1.171$ radians to 4 significant figures, where O is the origin.

(4)

The region shown shaded in Figure 3 is bounded by C_1 and C_2

(b) Find the perimeter of the shaded region, giving your answer to one decimal place.

(4)

(Total for Question 11 is 8 marks)

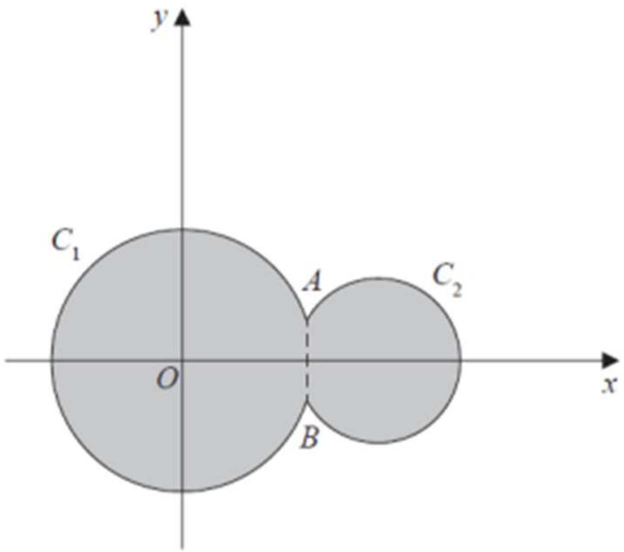


Figure 3

9. A circle with centre $A(3,-1)$ passes through the point $P(-9, 8)$ and the point $Q(15,-10)$

(a) Show that PQ is a diameter of the circle.

(2)

(b) Find an equation for the circle.

(3)

A point R also lies on the circle.
Given that the length of the chord PR is 20 units,

(c) find the length of the shortest distance from A to the chord PR.
Give your answer as a surd in its simplest form.

(2)

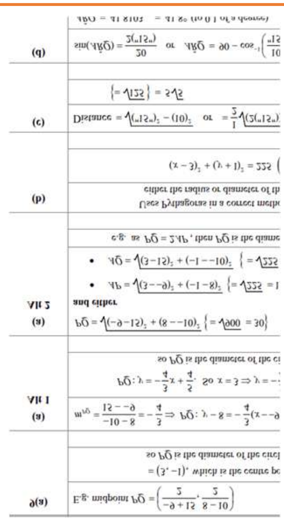
(d) Find the size of angle ARQ, giving your answer to the nearest 0.1 of a degree.

(2)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

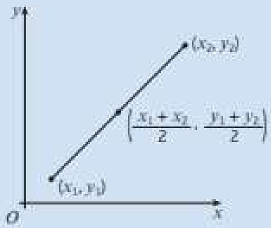
Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com



Summary of Key Points

Summary of key points

1 The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.



2 The perpendicular bisector of a line segment AB is the straight line that is perpendicular to AB and passes through the midpoint of AB.

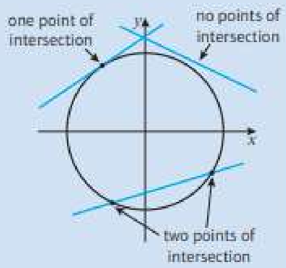
If the gradient of AB is m then the gradient of its perpendicular bisector, l, will be $-\frac{1}{m}$

3 The equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

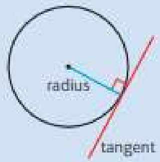
4 The equation of the circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

5 The equation of a circle can be given in the form: $x^2 + y^2 + 2fx + 2gy + c = 0$
This circle has centre $(-f, -g)$ and radius $\sqrt{f^2 + g^2 - c}$

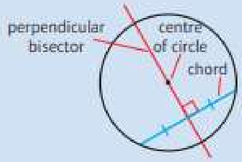
6 A straight line can intersect a circle once, by just touching the circle, or twice. Not all straight lines will intersect a given circle.



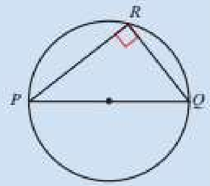
7 A tangent to a circle is perpendicular to the radius of the circle at the point of intersection.



8 The perpendicular bisector of a chord will go through the centre of a circle.

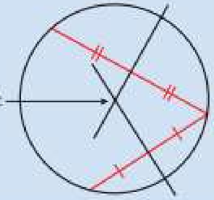


9
- If $\angle PRQ = 90°$ then R lies on the circle with diameter PQ.
- The angle in a semicircle is always a right angle.



10 To find the centre of a circle given any three points:
- Find the equations of the perpendicular bisectors of two different chords.
- Find the coordinates of intersection of the perpendicular bisectors.

Perpendicular bisectors intersect at the centre of the circle



T.133: mixed ex., P.53 : BSG