



KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS



KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12

Pure Mathematics

P1 12 Differentiation

Booklet

HGS Maths



Dr Frost Course



Name: _____

Class: _____

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Extract from Formulae booklet

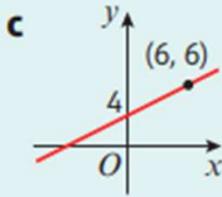
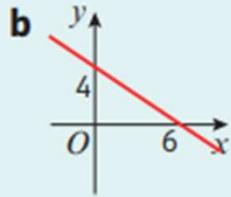
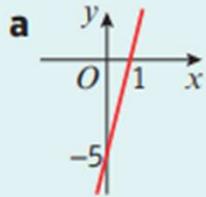
Past Paper Practice

Summary

Prior knowledge check

Prior knowledge check

1 Find the gradients of these lines.



← Section 5.1

2 Write each of these expressions in the form x^n where n is a positive or negative real number.

a $x^3 \times x^7$

b $\sqrt[3]{x^2}$

c $\frac{x^2 \times x^3}{x^6}$

d $\sqrt{\frac{x^2}{\sqrt{x}}}$

← Sections 1.1, 1.4

3 Find the equation of the straight line that passes through:

a $(0, -2)$ and $(6, 1)$ **b** $(3, 7)$ and $(9, 4)$

c $(10, 5)$ and $(-2, 8)$ ← Section 5.2

4 Find the equation of the perpendicular to the line $y = 2x - 5$ at the point $(2, 1)$.

← Section 5.3

12.3 Differentiating x^n

 If $y = ax^n$ then $\frac{dy}{dx} = nax^{n-1}$ (where a, n are constants)
i.e. multiply by the power and reduce the power by 1

The **operator** for differentiating is: $\frac{d}{dx}(\dots)$ - which means differentiate with respect to x whatever is in the bracket.

We also use $f(x)$ notation, i.e.

$$\text{If } y = f(x), \text{ then } \frac{dy}{dx} = f'(x)$$

i.e. $f'(x)$ is the derivative of $f(x)$

Notes

Worked Example

512e: Differentiate a collection of terms that require rewriting in the form ax^b

Find $\frac{dy}{dx}$ when $y = 2\sqrt[3]{x} + \frac{1}{\sqrt{x}} + \frac{2}{x^4}$

Worked Example

512f: Differentiate x^n to find the gradient of a tangent to a curve (i.e. the instantaneous rate of change) at a given point.

Find the gradient of the curve $y = 3x^2 - 5x + \frac{4}{\sqrt{x}}$ at the point $(4, 30)$.

Worked Example

Let $f(x) = 4x^2 - 8x + 3$

- a) Find the gradient of $y = f(x)$ at the point $(\frac{1}{2}, 0)$
- b) Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 8
- c) Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 4x - 5$

12.3 12.4 DFM worksheet

Question 1

Expand and simplify

$$\left(2 + \frac{\sqrt{x}}{4}\right)^2$$

expressing all terms in the form of ax^b

.....

Question 2

Write

$$\frac{\sqrt[3]{x}}{3} + \frac{5}{x} - \frac{3}{2x^3}$$

as an expression where all terms are in the form ax^b

.....

Question 3

Expand and simplify

$$(5 + 4\sqrt{x})^2$$

expressing all terms in the form of ax^b

.....

Question 4

Express

$$\frac{8\sqrt[3]{x} + 5x}{5x^3}$$

in the form of $ax^b + cx^d$

.....

Question 5

Express

$$\frac{4x + 3\sqrt[3]{x} + 7x^2}{6x^4}$$

in the form of $ax^b + cx^d + ex^f$

.....

Question 6

Express

$$\frac{10x^5 + 3x - 10\sqrt[3]{x}}{9\sqrt{x}}$$

in the form of $ax^b + cx^d + ex^f$

.....

Question 7

Find $f'(x)$ when $f(x) = x^7$

$f'(x) = \dots\dots\dots$

Question 8

Find $\frac{d}{dx}[-2x^8]$

$\frac{d}{dx}[-2x^8] = \dots\dots\dots$

Question 9

Find $f'(x)$ when $f(x) = \frac{1}{4}x^2$

$f'(x) = \dots\dots\dots$

Question 10

Differentiate the following expression with respect to t :

$2t^4 - 12t$

$\dots\dots\dots$

Question 11

Differentiate the following expression with respect to t :

$-9t^5 + 10t^3 + 10t^2 + 12 - 4t^{-2}$

$\dots\dots\dots$

Question 12

Differentiate the following expression with respect to x :

$7x^4 + 2x^3$

$\dots\dots\dots$

Question 13

Find $\frac{dy}{dx}$ when $y = 3\sqrt{x} - \frac{2}{x^4}$

$\frac{dy}{dx} = \dots\dots\dots$

Question 14

Find $\frac{d}{dx}\left[-\frac{4}{x} + \frac{3}{x^8}\right]$

$\frac{d}{dx}[\dots] = \dots\dots\dots$

Question 15

Find $f'(x)$ when $f(x) = 4x + \sqrt[3]{x} + 3\sqrt{x} + \frac{5}{x^5}$

$f'(x) = \dots\dots\dots$

Question 16

Find the gradient of the curve $y = \frac{1}{\sqrt{x}} + \frac{1}{x^2}$ at the point (1, 2).

Gradient =

Question 17

Find the gradient of the curve $y = x^3 - 3x^2 + 5x$ at the point (5, 75).

Gradient =

Question 18

Find the gradient of the curve $y = 2x^3 + x + 5$ at the point (1, 8).

Gradient =

12.5 Differentiating Functions with Two or More Terms

Notes

Worked Example

512i: Differentiate a polynomial expression written as a fraction with a single term on the denominator.

Find $\frac{dy}{dx}$ when

$$y = \frac{5x^{-\frac{1}{2}} + 2x^4}{3x^3}$$

$$\frac{dy}{dx} = \text{✎ } \boxed{}$$

512j: Differentiate an expression written as a fraction with a single term on the denominator and that contains a root.

Find $f'(x)$ when

$$f(x) = \frac{3x^5 + 4x^4 + 4x}{2\sqrt{x}}$$

$$f'(x) = \text{✎ } \boxed{}$$

Worked Example

512h: Determine the value of a constant given the gradient of the tangent to a curve at a point.

A curve has equation $y = f(x)$, where

$$f(x) = kx^2 + 6x - 5$$

Given that

$$f'(-2) = -14$$

find the value of k .

512k: Differentiate an expression written as a fraction with a single term on the denominator and that contains a root.

Find $f'(x)$ when

$$f(x) = \frac{(2x^2 - 3)^2}{2\sqrt[3]{x}}$$

$f'(x) =$

16. The curve C has equation $y = f(x)$ where

$$f(x) = ax^3 + 15x^2 - 39x + b$$

and a and b are constants.

Given

- the point $(2, 10)$ lies on C
- the gradient of the curve at $(2, 10)$ is -3

(a) (i) show that the value of a is -2

(ii) find the value of b .

(4)

(b) Hence show that C has no stationary points.

(3)

(c) Write $f(x)$ in the form $(x - 4)Q(x)$ where $Q(x)$ is a quadratic expression to be found.

(2)

(d) Hence deduce the coordinates of the points of intersection of the curve with equation

$$y = f(0.2x)$$

and the coordinate axes.

(2)

Your Turn

16. The curve C has equation $y = f(x)$ where

$$f(x) = ax^3 + 17x^2 - 17x + b$$

and a and b are constants.

Given

- the point $(2, 8)$ lies on C
- the gradient of the curve at $(2, 8)$ is 3

(a) (i) show that the value of a is -4

(ii) find the value of b .

(4)

(b) Hence show that C has two stationary points.

(3)

(c) Write $f(x)$ in the form $(x - 3)Q(x)$ where $Q(x)$ is a quadratic expression to be found.

(2)

(d) Hence deduce the coordinates of the points of intersection of the curve with equation

$$y = f(0.2x)$$

and the coordinate axes.

(2)

(Total for Question 16 is 11 marks)

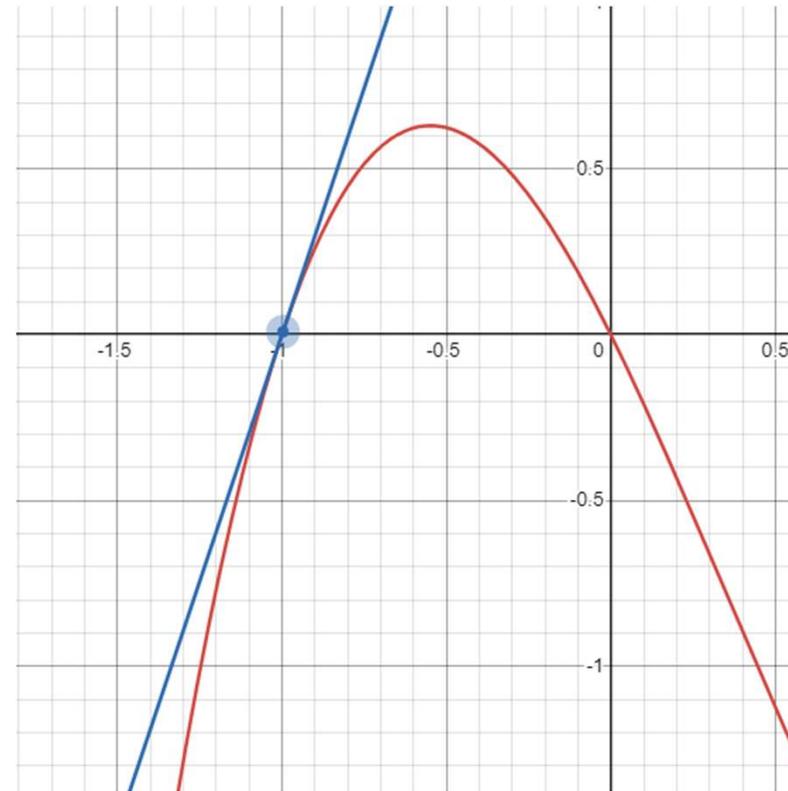
12.6 Gradients, Tangents and Normals

$\frac{dy}{dx}$ evaluated at a given point $P(x, y)$ gives the value of the tangent at P

I.e. for the equation of the tangent:

$$y - y_1 = m(x - x_1)$$

$$m = \frac{dy}{dx} \text{ evaluated at } x_1$$



**for the normal use the negative reciprocal*

On your calculator this is written as:

$$\frac{d}{dx} (\quad) \Big|_{x=}$$

Notes

Worked Example

520c: Find an equation of a tangent to a polynomial curve.

Find an equation of the tangent to the curve
 $y = x^2 + x + 1$ at the point $(4, 21)$.

Worked Example

520d: Find an equation of a normal to a polynomial curve.

Find an equation of the normal to the curve $y = 2x^2 + 4$ at the point $(5, 54)$.

Worked Example

Find the equation of the normal to the curve with equation $y = 8 - 3\sqrt{x}$ at the point where $x = 4$.
Give your answer in the form $ax + by + c = 0$

Worked Example

520f: Determine the coordinate of a point on a curve given the equation of a line parallel to the tangent or normal.

A curve $y = f(x)$ has the equation

$$f(x) = 3x^3 + 18x^2 + 41x + 33.$$

The tangent at point P is parallel to the line $5y - 25x = 27$

.

Find the coordinates of P .

Worked Example

520g: Determine the equation of a tangent or normal to a curve where the function involves fractional or negative powers.

The curve $y = f(x)$ is given by $f(x) = 3x^3 + \frac{6}{\sqrt{x}} - 40$

The point P lies on the curve and has coordinates $(1, -31)$.

Find an equation of the tangent to the curve at P .

Worked Example

520h: Determine the equation of the tangent or normal of a curve at an intercept with the axes.

The curve $y = f(x)$ is given by the equation

$$f(x) = 7x^3 + 6x^2 - 6x + 26$$

Find an equation of the normal to the curve at the point where it intercepts the y -axis.

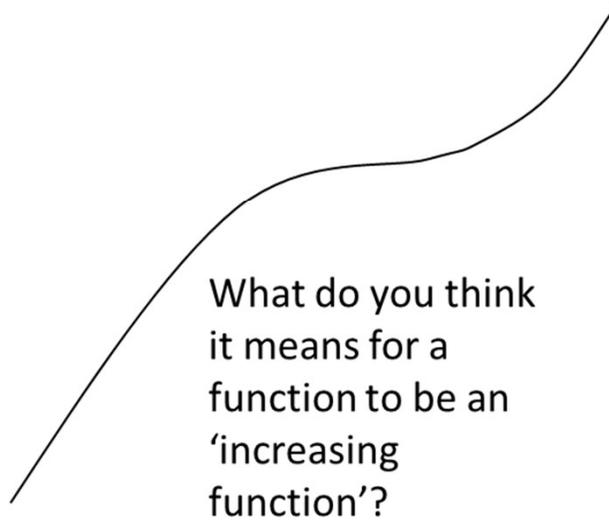
Worked Example

The point P with x -coordinate $\frac{1}{2}$ lies on the curve with equation $y = 4x^2$.

The normal to the curve at P intersects the curve at points P and Q .

Find the coordinates of Q

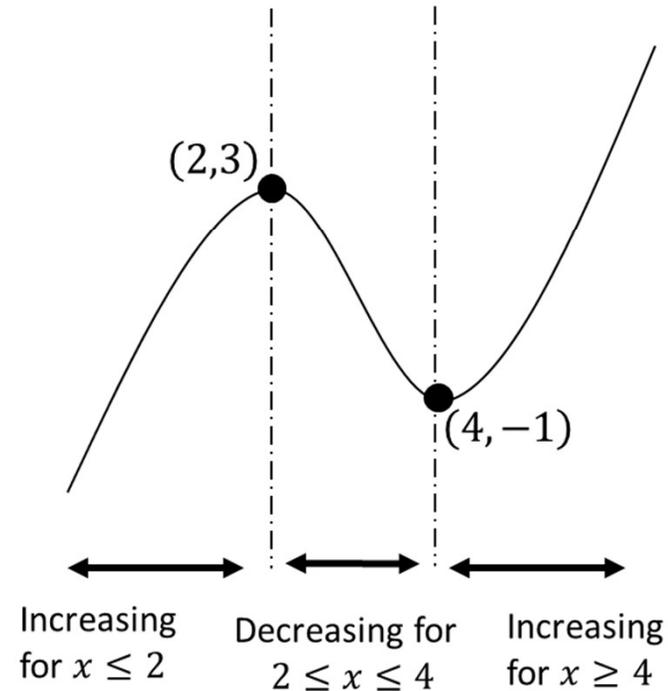
12.7 Increasing and Decreasing Functions



An increasing function is one whose gradient is always at least 0.
 $f'(x) \geq 0$ for all x .

It would be '**strictly increasing**' if $f'(x) > 0$ for all x , i.e. is not allowed to go horizontal.

A function can also be increasing and decreasing in certain intervals.



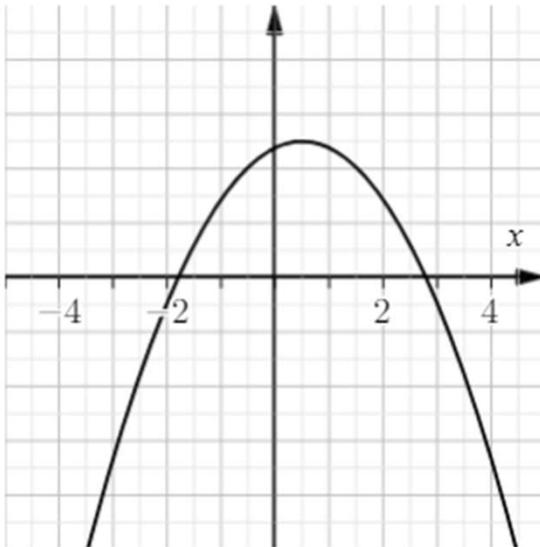
We could also write " $f(x)$ is decreasing in the interval $[2,4]$ "
 $[a, b]$ represents all the real numbers between a and b inclusive, i.e:
 $[a, b] = \{x : a \leq x \leq b\}$

Notes

Worked Example

516a: Understand the terms increasing and decreasing and apply to a drawn graph.

A sketch of the curve $y = f(x)$ is shown below.

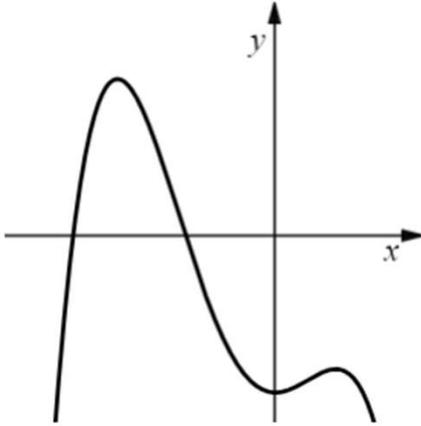


Using the graph, find the interval in which $f(x)$ is decreasing.

Worked Example

516b: Determine the interval for which a quadratic function is increasing or decreasing.

A sketch of the curve $y = f(x)$ is shown below.



The curve has the gradient function

$$\frac{dy}{dx} = -12x^3 - 13x^2 + 14x.$$

Find the interval in which the curve is increasing.

Worked Example

516d: Determine the interval for which a cubic function is increasing or decreasing.

Find the interval on which the function $y = 3x^3 + 12x^2 - 9x - 8$ is increasing.

Worked Example

516f: Determine the range of values for which a function is increasing/decreasing involving a reciprocal term.

A function is given by the equation

$$f(x) = -3x - \frac{32}{x} - 20, \quad x \neq 0$$

Find the exact interval on which $f(x)$ is decreasing.

Worked Example

Show that the function

$f(x) = x^3 + 6x^2 + 21x + 2$ is increasing for all real values of x .

Worked Example

Show that the function $3 + 4x(-x^2 - 5)$ is decreasing for all $x \in \mathbb{R}$

5. A curve has equation

$$y = 3x^2 + \frac{24}{x} + 2 \quad x > 0$$

(a) Find, in simplest form, $\frac{dy}{dx}$

(3)

(b) Hence find the exact range of values of x for which the curve is increasing.

(2)

Your Turn

5.

A curve has equation

$$y = \sqrt{x} + \frac{4}{x} + 4, \quad x > 0$$

(a) Find, in simplest form $\frac{dy}{dx}$ (3)

(b) Hence find the range of values of x for which the curve is increasing (2)

1. A curve has equation

$$y = \frac{2}{3}x^3 - \frac{7}{2}x^2 - 4x + 5$$

(a) Find $\frac{dy}{dx}$ writing your answer in simplest form.

(2)

(b) Hence find the range of values of x for which y is decreasing.

(4)

Your Turn

1. A curve has equation

$$y = \frac{5}{3}x^3 + \frac{13}{2}x^2 - 6x + 114$$

(a) Find $\frac{dy}{dx}$ writing your answer in simplest form.

(2)

(b) Hence find the range of values of x for which y is decreasing.

(4)

(Total for Question 1 is 6 marks)

12.8 Second Order Derivatives

When you differentiate once, the expression you get is known as the **first derivative**. Unsurprisingly, when we differentiate a second time, the resulting expression is known as the **second derivative**. And so on...

Original Function

First Derivative

Second Derivative

$$\begin{array}{l} y = x^4 \end{array} \begin{array}{l} \nearrow \text{Leibniz's} \\ \rightarrow \\ \searrow \text{Newton's} \end{array} \begin{array}{l} \frac{dy}{dx} = 4x^3 \\ y' = 4x^3 \\ \dot{y} = 4x^3 \end{array} \begin{array}{l} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \begin{array}{l} \frac{d^2y}{dx^2} = 12x^2 \\ y'' = 12x^2 \\ \ddot{y} = 12x^2 \end{array}$$

$$\begin{array}{l} f(x) = x^4 \end{array} \begin{array}{l} \xrightarrow{\text{Lagrange's}} \\ \rightarrow \\ \rightarrow \end{array} \begin{array}{l} f'(x) = 4x^3 \\ f''(x) = 12x^2 \end{array}$$

You can similarly have the third derivative ($\frac{d^3y}{dx^3}$), although this is no longer in the A Level syllabus. We'll see why might use the second derivative soon...

Notes

Notes

Worked Example

515a: Determine the second derivative of a function.

A function is given by the equation

$$y = (4x^5 + 3x^2)^2$$

Find an expression for $\frac{d^2y}{dx^2}$.

Worked Example

If $f(x) = 3\sqrt{x} + \frac{1}{2\sqrt{x}}$, find $f''(x)$

Worked Example

515b: Evaluate the second derivative of a function at a specific point.

A function is given by the equation

$$f(x) = \frac{3x^{10} + 2x^5}{x^6}$$

Determine $f''(-4)$.

Give your answer to three significant figures.

14. A curve has equation $y = g(x)$.

Given that

- $g(x)$ is a cubic expression in which the coefficient of x^3 is equal to the coefficient of x
- the curve with equation $y = g(x)$ passes through the origin
- the curve with equation $y = g(x)$ has a stationary point at $(2, 9)$

(a) find $g(x)$,

(7)

(b) prove that the stationary point at $(2, 9)$ is a maximum.

(2)

Your Turn

14 A curve has equation $y = g(x)$.

Given that

- $g(x)$ is a cubic expression in which the coefficient of x^3 is equal to the coefficient of x
- the curve with equation $y = g(x)$ passes through the origin
- the curve with equation $y = g(x)$ has a stationary point at $(3, 18)$

(a) find $g(x)$,

(7)

(b) prove that the stationary point at $(3, 18)$ is a maximum.

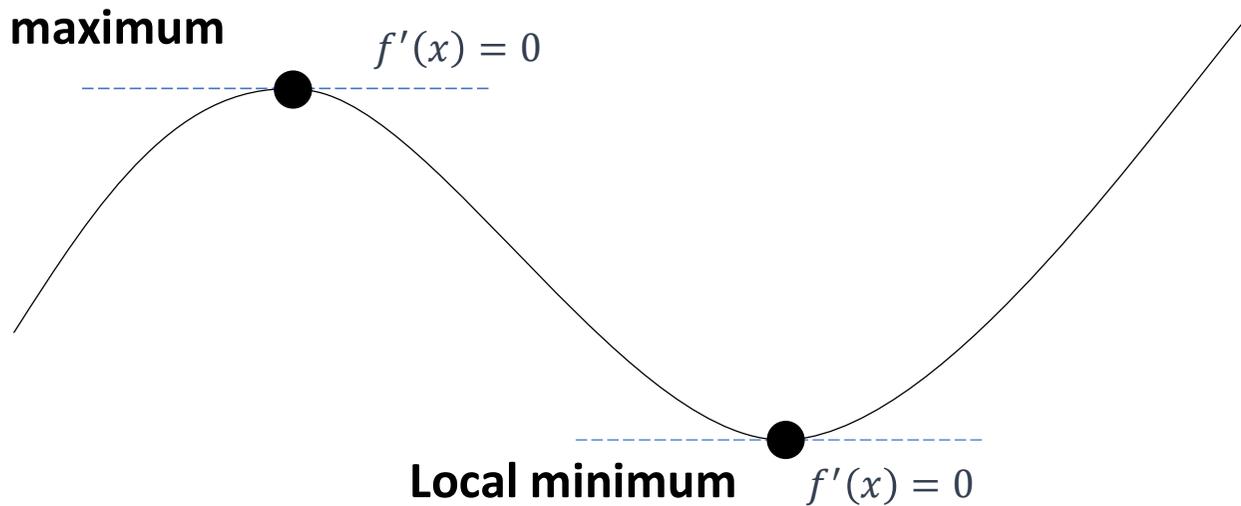
(2)

(Total for Question 14 is 9 marks)

12.9 Stationary Points

A stationary point is where the gradient is 0, i.e. $f'(x) = 0$.

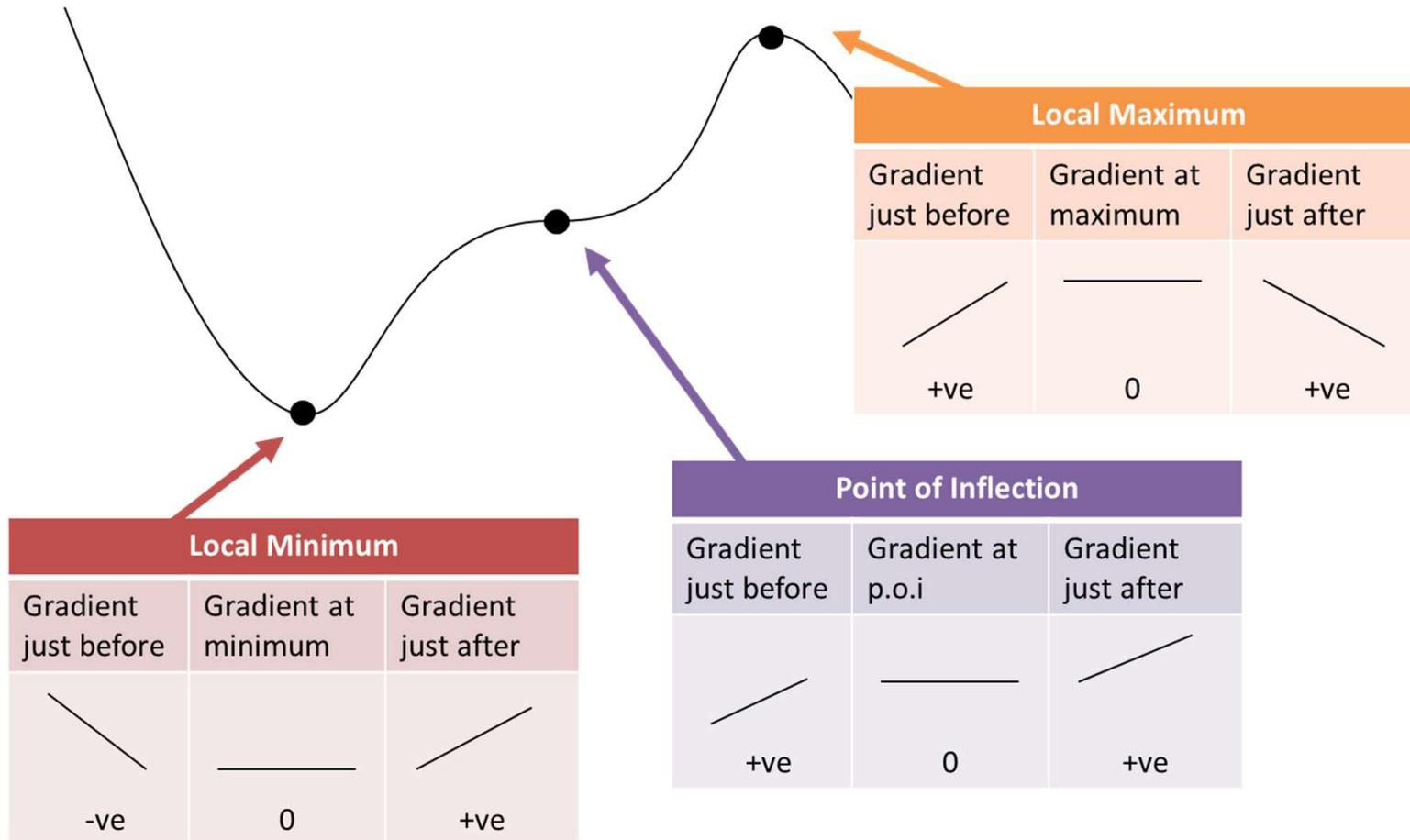
Local maximum



Note: It's called a '**local**' maximum because it's the function's largest output within the vicinity. Functions may also have a '**global**' maximum, i.e. the maximum output across the entire function. This particular function doesn't have a global maximum because the output keeps increasing up to infinity. It similarly has no global minimum, as with all cubics.

How do we tell what type of stationary point?

Method 1: Look at gradient just before and just after point.



Using the second derivative

At a stationary point $x = a$:

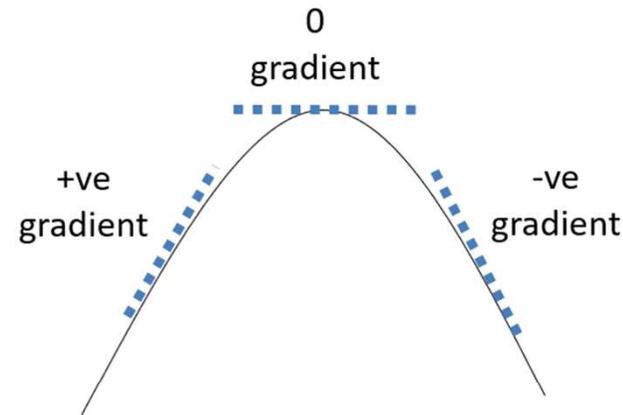
- If $f''(a) > 0$ the point is a local minimum.
- If $f''(a) < 0$ the point is a local maximum.
- If $f''(a) = 0$ it could be any type of point, so resort to Method 1.

Recall the gradient gives a measure of the **rate of change** of y , i.e. how much the y value changes as x changes.

Thus by differentiating the gradient function, the **second derivative tells us the rate at what the gradient is changing**.

Thus if the second derivative is positive, the gradient is increasing.

If the second derivative is negative, the gradient is decreasing.



At a maximum point, we can see that as x increases, the gradient is decreasing from a positive value to a negative value.

$$\therefore \frac{d^2y}{dx^2} < 0$$

Worked Example

Find the least value of

$$f(x) = x^2 - 4x + 9$$

Worked Example

Find the turning point of

$$y = \sqrt{x} - x$$

Worked Example

Find the stationary point on the curve with equation

$y = x^4 - 32x$, and determine whether it is a local maximum, a local minimum or a point of inflection.

Worked Example

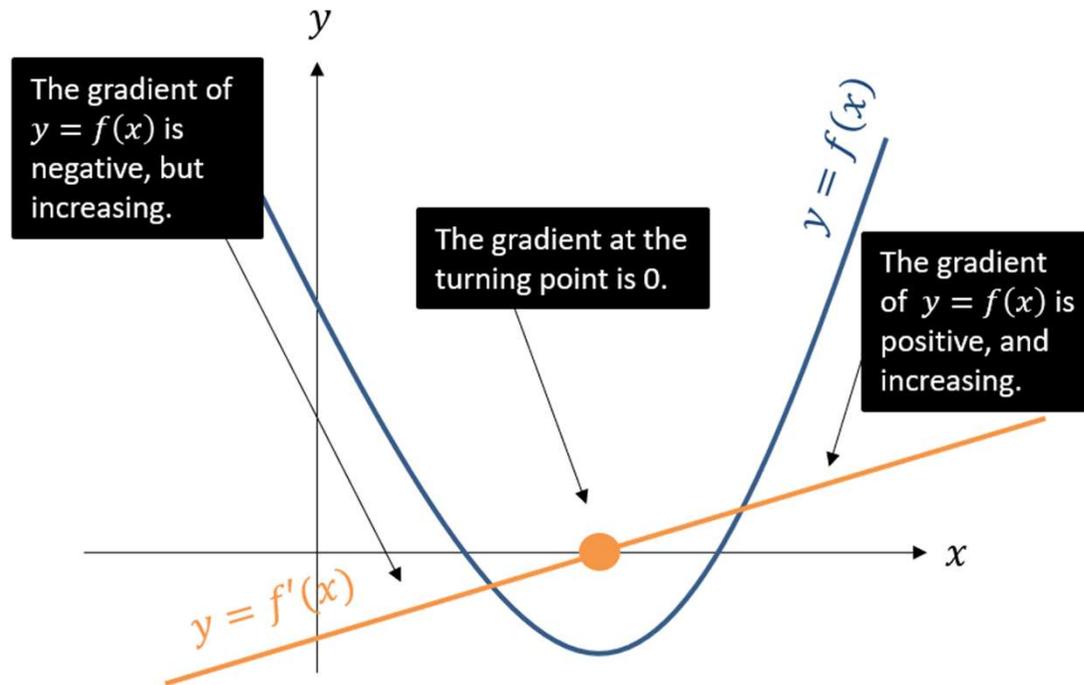
Find the coordinates of the stationary points on the curve with equation $y = 2x^3 - 15x^2 + 24x + 6$ and use the second derivative to determine their nature.

12.10 Sketching Gradient Functions

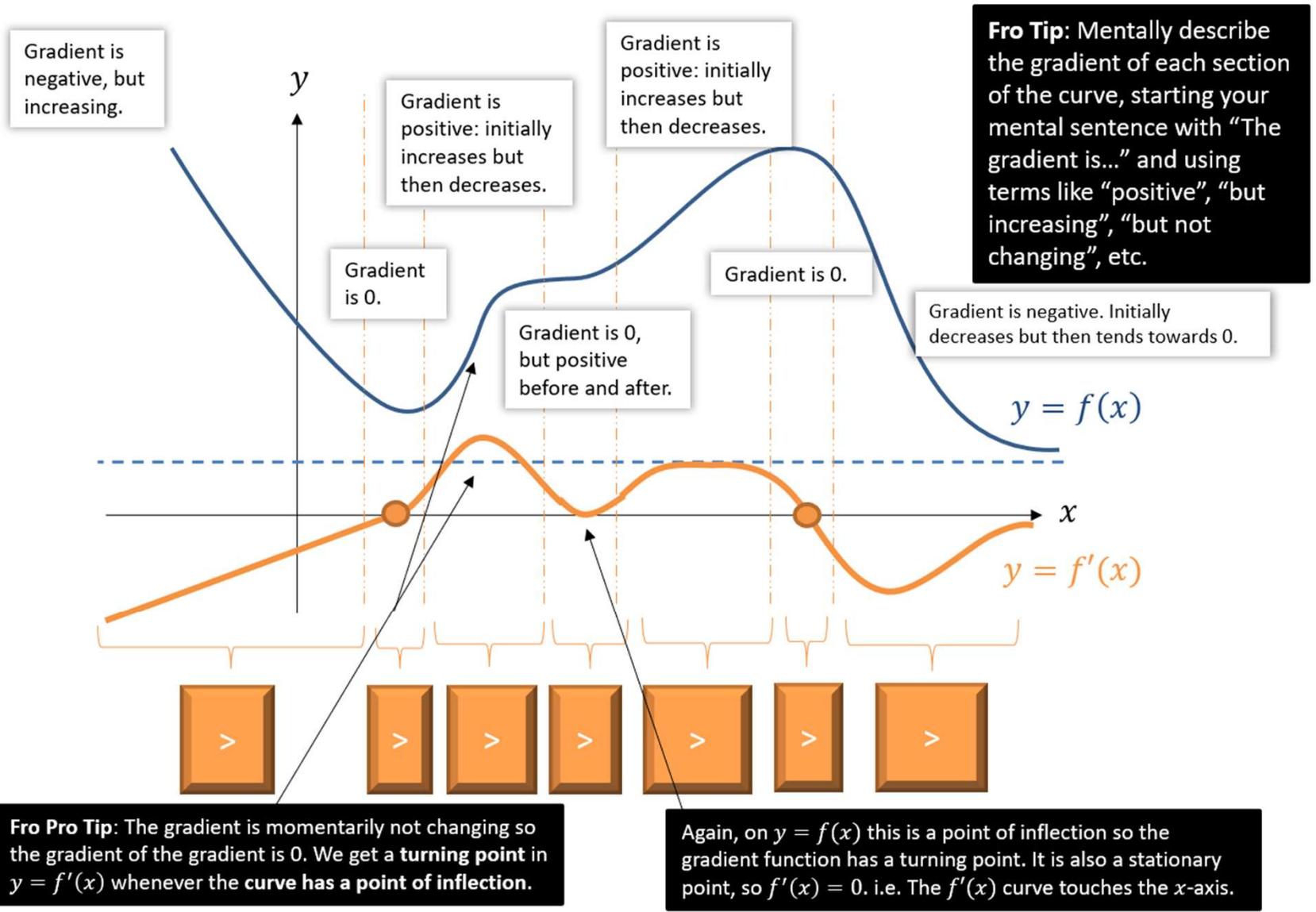
The new A Level specification specifically mentions being able to sketch $y = f'(x)$.

If you know the function $f'(x)$ explicitly (e.g. because you differentiated $y = f(x)$), you can use your knowledge of sketching straight line/quadratic/cubic graphs.

But in other cases **you won't be given the function explicitly, but just the sketch.**



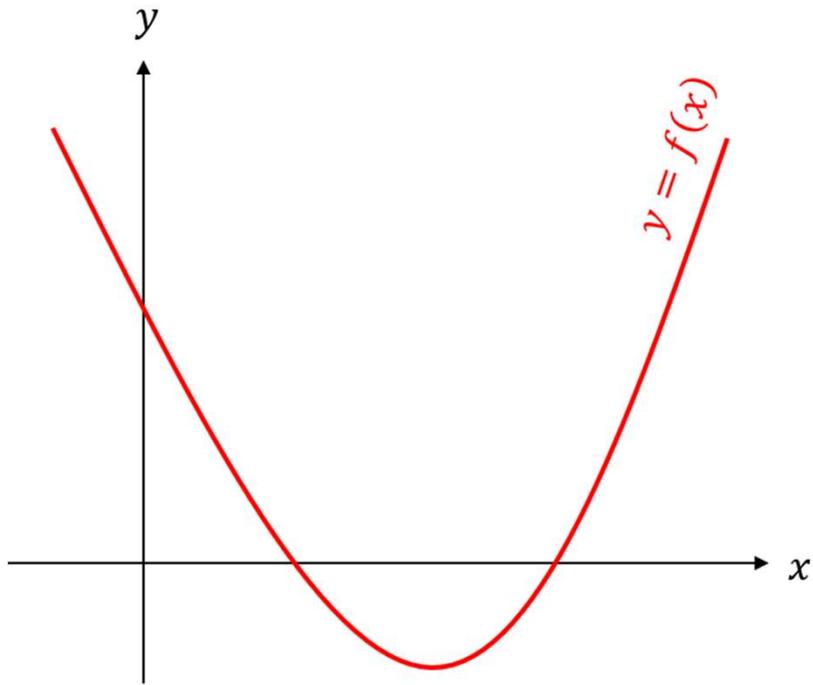
Notes



Notes

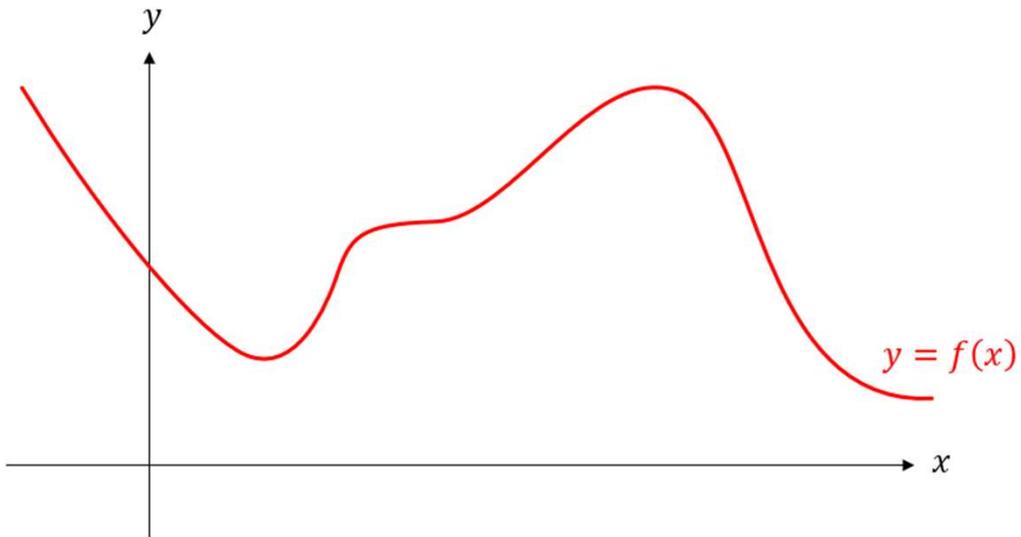
Worked Example

Sketch $y = f'(x)$ on the same axes



Worked Example

Sketch $y = f'(x)$ on the same axes



Worked Example

A positive cubic has the equation $y = f(x)$.

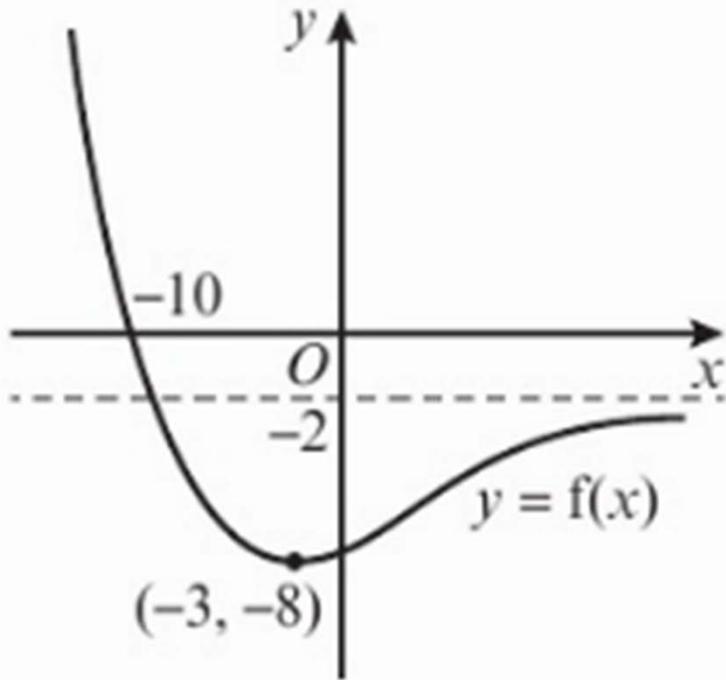
The curve has stationary points at $(-1, 4)$ and $(1, 0)$ and cuts the x -axis at $(-3, 0)$.

Sketch the gradient function, $y = f'(x)$, showing the coordinates of any points where the curve cuts or meets the x -axis.

Worked Example

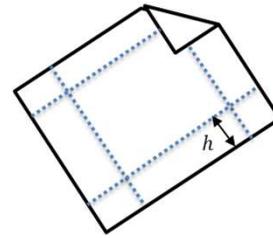
The diagram shows the curve with equation $y = f(x)$. The curve has an asymptote at $y = -2$ and a turning point at $(-3, -8)$. It cuts the x -axis at $(-10, 0)$.

- Sketch the graph of $y = f'(x)$.
- State the equation of the asymptote of $y = f'(x)$.



12.11 Modelling with Differentiation

1. Optimisation Problems/Modelling, *e.g.*

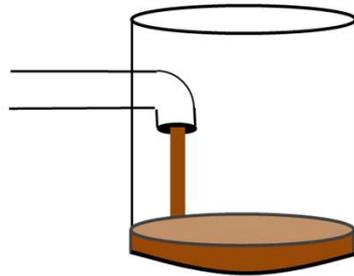


We have a sheet of A4 paper, which we want to fold into a cuboid. What height should we choose for the cuboid in order to maximise the volume?

2. Rate of change:

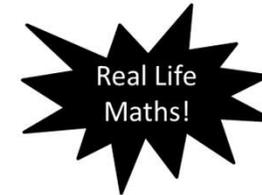
Up to now we've had y in terms of x , where $\frac{dy}{dx}$ means "the **rate** at which y changes with respect to x ".

But we can use similar gradient function notation for other **physical quantities**.



A sewage container fills at a rate of 20 cm^3 per second.

How could we use appropriate notation to represent this?



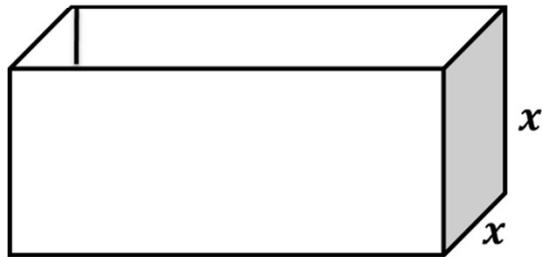
$$\frac{dV}{dt} = 20 \text{ cm}^3 / \text{s}$$

"The rate at which the volume V changes with respect to time t ."

Example Optimisation Problem

Optimisation problems in an exam usually follow the following pattern:

- There are 2 variables involved (you may have to introduce one yourself), typically lengths.
- There are expressions for **two different physical quantities**:
 - One is a **constraint**, e.g. "the surface area is 20cm²".
 - The **other we wish to maximise/minimise**, e.g. "we wish to maximise the volume".
- We use the constraint to **eliminate one of the variables** in the latter equation, so that it is then **just in terms of one variable**, and we can then use differentiation to find the turning point.



[Textbook] A large tank in the shape of a cuboid is to be made from 54m² of sheet metal. The tank has a horizontal base and no top. The height of the tank is x metres. Two of the opposite vertical faces are squares.

a) Show that the volume, V m³, of the tank is given by

$$V = 18x - \frac{2}{3}x^3.$$

We need to introduce a second variable ourselves so that we can find expressions for the surface area and volume.

$$2x^2 + 3xy = 54$$

$$V = x^2y$$

But we want V just in terms of x :

$$y = \frac{54 - 2x^2}{3x} \rightarrow V = x^2 \left(\frac{54 - 2x^2}{3x} \right)$$

$$V = \frac{54x^2 - 2x^4}{3x} = 18x - \frac{2}{3}x^3$$

These are the two equations mentioned in the guidance: one for surface area and one for volume.

b) Given that x can vary, use differentiation to find the maximum or minimum value of V .

$$\frac{dV}{dx} = 18 - 2x^2 = 0 \quad \therefore x = 3$$

$$V = 18(3) - \frac{2}{3}(3)^3 = 36$$

Once we have the 'optimal' value of x , we sub it back into V to get the best possible volume.

Notes

Worked Example

Given that the volume, $V \text{ cm}^3$, of an expanding sphere is related to its radius, $r \text{ cm}$, by the formula $V = \frac{4}{3}\pi r^3$, find the rate of change of volume with respect to radius at the instant when the radius is 5 cm .

Worked Example

A cuboid is to be made from 54m^2 of sheet metal.

The cuboid has a horizontal base and no top.

The height of the cuboid is x metres.

Two of the opposite vertical faces are squares.

- a) Show that the volume, $V \text{ m}^3$, of the tank is given by $V = 18x - \frac{2}{3}x^3$.
- b) Given that x can vary, use differentiation to find the maximum or minimum value of V .
- c) Justify that the value of V you have found is a maximum

AS 2018

8. A lorry is driven between London and Newcastle.

In a simple model, the cost of the journey £ C when the lorry is driven at a steady speed of v kilometres per hour is

$$C = \frac{1500}{v} + \frac{2v}{11} + 60$$

(a) Find, according to this model,

(i) the value of v that minimises the cost of the journey,

(ii) the minimum cost of the journey.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(6)

(b) Prove by using $\frac{d^2C}{dv^2}$ that the cost is minimised at the speed found in (a)(i).

(2)

(c) State one limitation of this model.

(1)

Your Turn

8. A lorry is driven between London and Newcastle.

In a simple model, the cost of the journey £ C when the lorry is driven at a steady speed of v kilometres per hour is

$$C = \frac{1400}{v} + \frac{3v}{13} + 50$$

- (a) Find, according to this model,
- (i) the exact value of v that minimises the cost of the journey,
 - (ii) the minimum cost of the journey.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(6)

- (b) Prove, by using $\frac{d^2C}{dv^2}$, that the cost is minimised at the speed found in part (a)(i).

(2)

- (c) State one limitation of this model.

(1)

12. A company makes drinks containers out of metal.

The containers are modelled as closed cylinders with base radius r cm and height h cm and the capacity of each container is 355 cm^3

The metal used

- for the circular base and the curved side costs 0.04 pence/cm^2
- for the circular top costs 0.09 pence/cm^2

Both metals used are of negligible thickness.

(a) Show that the total cost, C pence, of the metal for one container is given by

$$C = 0.13\pi r^2 + \frac{28.4}{r} \quad (4)$$

(b) Use calculus to find the value of r for which C is a minimum, giving your answer to 3 significant figures.

(4)

(c) Using $\frac{d^2C}{dr^2}$ prove that the cost is minimised for the value of r found in part (b).

(2)

(d) Hence find the minimum value of C , giving your answer to the nearest integer.

(2)

Your Turn

12. A company makes paint containers out of metal.

The containers are modelled as closed cylinders with base radius r cm and height h cm and the capacity of each container is 7800 cm^3

The metal used

- for the circular base and the curved side costs 0.05 pence/cm^2
- for the circular top costs 0.08 pence/cm^2

Both metals used are of negligible thickness.

- (a) Show that the total cost, C pence, of the metal for one container is given by

$$C = 0.13\pi r^2 + \frac{780}{r}$$

(4)

- (b) Use calculus to find the value of r for which C is a minimum, giving your answer to 3 significant figures.

(4)

- (c) Using $\frac{d^2 C}{dr^2}$ prove that the cost is minimised for the value of r found in part (b).

(2)

- (d) Hence find the minimum value of C , giving your answer to the nearest integer.

(2)

(Total for Question 12 is 12 marks)

12.2 Finding the Derivative

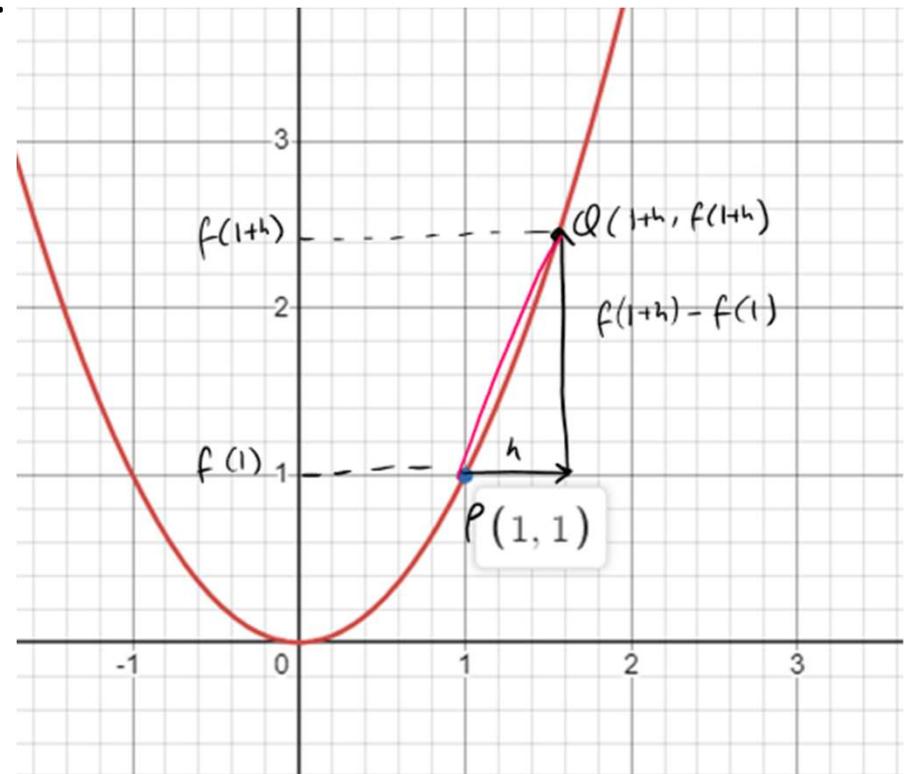
Where does the rule/method we have been using come from?

At GCSE you learnt how to *estimate* the gradient of a curve by *sketching* the tangent and then finding the gradient of the tangent at that point.

We can generalise this idea to the gradient of a *chord* from the point $P(x, f(x))$ we are interested in to a point some distance away $Q(x + h, f(x + h))$, i.e.

So the gradient of the chord PQ is

$$\frac{f(x + h) - f(x)}{h}$$



Notes

The gradient of this chord is not the actual gradient at P (*i.e. the gradient of the tangent to $y = f(x)$ at P*). It is an approximation.

The approximation will get better as the distance between P and Q reduces, *i.e. h gets smaller, which we write as $h \rightarrow 0$ (read as h tends to zero)*

Therefore, you need to know and use:

The gradient function, or derivative, of the curve $y = f(x)$ is written as $f'(x)$ or $\frac{dy}{dx}$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The gradient function can be used to find the gradient of the curve for any value of x .

Practically you don't have to use the $\lim_{h \rightarrow 0}$ bit, just use $h \rightarrow 0$. The above is just how it is stated in formulae books/textbooks.

Notes

Worked Example

The point A with coordinates $(4, 16)$ lies on the curve with equation $y = x^2$.

At point A the curve has gradient g .

a) Show that $g = \lim_{h \rightarrow 0} (8 + h)$

b) Deduce the value of g .

Worked Example

Prove from first principles that the derivative of $5x$ is 5

Worked Example

Prove from first principles that the derivative of $5x^2$ is $10x$

Worked Example

Prove from first principles that the derivative of x^3 is $3x^2$

5.

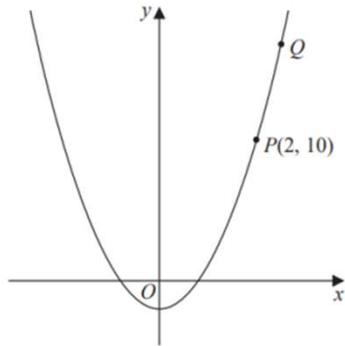


Figure 1

Figure 1 shows part of the curve with equation $y = 3x^2 - 2$

The point $P(2, 10)$ lies on the curve.

(a) Find the gradient of the tangent to the curve at P . (2)

The point Q with x coordinate $2 + h$ also lies on the curve.

(b) Find the gradient of the line PQ , giving your answer in terms of h in simplest form. (3)

(c) Explain briefly the relationship between part (b) and the answer to part (a). (1)

Your Turn

5.

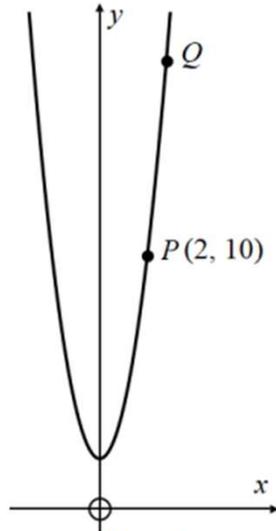


Figure 1

Figure 1 shows part of the curve with equation $y = 2x^2 + 2$

The point $P(2, 10)$ lies on the curve.

(a) Find the gradient of the tangent to the curve at P . (2)

The point Q with x coordinate $2 + h$ also lies on the curve.

(b) Find the gradient of the line PQ , giving your answer in terms of h in simplest form. (3)

(c) Explain briefly the relationship between part (b) and the answer to part (a). (1)

(Total for Question 5 is 6 marks)

Differentiation

First Principles

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Past Paper Questions

1. The curve C has equation

$$y = 3x^4 - 8x^3 - 3$$

(a) Find (i) $\frac{dy}{dx}$

(ii) $\frac{d^2y}{dx^2}$

(3)

(b) Verify that C has a stationary point when $x = 2$

(2)

(c) Determine the nature of this stationary point, giving a reason for your answer.

(2)



Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

		Δ marks	
(c)	$\frac{d^2y}{dx^2} = 48 > 0$ and states "hence the stationary point is a minimum"	VIH	5
	Substitutes $x = 2$ into $\frac{d^2y}{dx^2} = 3(2)^3 - 48(2)$	MI	1/1P
(b)	Shows $\frac{dy}{dx} = 0$ and states "hence there is a stationary point"	VI	5
	Substitutes $x = 2$ into $\frac{dy}{dx} = 12(2)^3 - 24(2)$	MI	1/1P
1(a)	(ii) $\frac{d^2y}{dx^2} = 24x^2 - 48x$	VIH	1/1P
	(i) $\frac{dy}{dx} = 12x^3 - 24x^2$	VI	1/1P

Summary of Key Points

1 The **gradient** of a **curve** at a given point is defined as the gradient of the **tangent** to the curve at that point.

2 The **gradient function**, or **derivative**, of the curve $y = f(x)$ is written as $f'(x)$ or $\frac{dy}{dx}$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The gradient function can be used to find the gradient of the curve for any value of x .

3 For all real values of n , and for a constant a :

- If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ • If $f(x) = ax^n$ then $f'(x) = anx^{n-1}$
- If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$ • If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$

4 For the quadratic curve with equation $y = ax^2 + bx + c$, the derivative is given by

$$\frac{dy}{dx} = 2ax + b$$

5 If $y = f(x) \pm g(x)$, then $\frac{dy}{dx} = f'(x) \pm g'(x)$.

6 The tangent to the curve $y = f(x)$ at the point with coordinates $(a, f(a))$ has equation

$$y - f(a) = f'(a)(x - a)$$

7 The normal to the curve $y = f(x)$ at the point with coordinates $(a, f(a))$ has equation

$$y - f(a) = -\frac{1}{f'(a)}(x - a)$$

8 • The function $f(x)$ is **increasing** on the interval $[a, b]$ if $f'(x) \geq 0$ for all values of x such that $a < x < b$.

• The function $f(x)$ is **decreasing** on the interval $[a, b]$ if $f'(x) \leq 0$ for all values of x such that $a < x < b$.

9 Differentiating a function $y = f(x)$ twice gives you the second order derivative, $f''(x)$ or $\frac{d^2y}{dx^2}$

10 Any point on the curve $y = f(x)$ where $f'(x) = 0$ is called a **stationary point**. For a small positive value h :

Type of stationary point	$f'(x - h)$	$f'(x)$	$f'(x + h)$
Local maximum	Positive	0	Negative
Local minimum	Negative	0	Positive
Point of inflection	Negative	0	Negative
	Positive	0	Positive

11 If a function $f(x)$ has a stationary point when $x = a$, then:

- if $f''(a) > 0$, the point is a local minimum
- if $f''(a) < 0$, the point is a local maximum.

If $f''(a) = 0$, the point could be a local minimum, a local maximum or a point of inflection. You will need to look at points on either side to determine its nature.