KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Pure Mathematics
P1 3 Equations and Inequalities Booklet

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

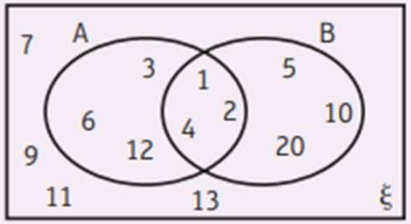
Prior knowledge check

Prior knowledge check

1 A = {factors of 12}

B = {factors of 20}

Write down the numbers in each of these sets:



a $A \cap B$ **b** $(A \cup B)'$

← **GCSE Mathematics**

2 Simplify these expressions.

a $\sqrt{75}$ **b** $\dfrac{2\sqrt{45} + 3\sqrt{32}}{6}$

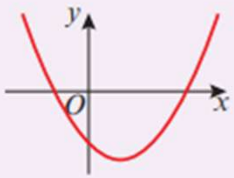
← **Section 1.5**

3 Match the equations to the correct graph. Label the points of intersection with the axes and the coordinates of the turning point.

a $y = 9 - x^2$ **b** $y = (x - 2)^2 + 4$

c $y = (x - 7)(2x + 5)$

i



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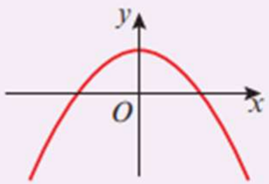
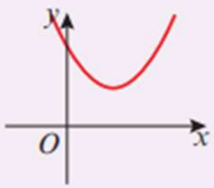


iii



← **Section 2.4**

3.2) Quadratic simultaneous equations

The solution(s) to an equation may be:

A single value: $2x + 1 = 5$

Multiple values: $x^2 + 3x + 2 = 0$

An infinitely large set of values: $x > 3$

No (real) values! $x^2 = -1$

Every value! $x^2 + x = x(x + 1)$

The point is that you shouldn't think of the solution to an equation/inequality as an 'answer', but a **set** of values, which might just be a set of 1 value (known as a singleton set), a set of no values (i.e. the empty set $\emptyset$), or an infinite set (in the last example above, this was $\mathbb{R}$)

✎ The solutions to an equation are known as the **solution set**.

Solutions sets

For simultaneous equations, the same is true, except each 'solution' in the solution set is an assignment to **multiple** variables.
All equations have to be satisfied **at the same time, i.e. 'simultaneously'**.

Scenario	**Example**	**Solution Set**
A single solution:	$x + y = 9$ $x - y = 1$	**Solution 1:** $\boldsymbol{x = 5, y = 4}$ To be precise here, the solution set is of size 1, but this solution is an assignment to multiple variables, i.e. a pair of values.
Two solutions:	$x^2 + y^2 = 10$ $x + y = 4$	**Solution 1:** $\boldsymbol{x = 3, y = 1}$ **Solution 2:** $\boldsymbol{x = 1, y = 3}$ This time we have two solutions, each an x, y pair.
No solutions:	$x + y = 1$ $x + y = 3$	The solution set is empty, i.e. $\emptyset$, as both equation can't be satisfied at the same time.
Infinitely large set of solutions:	$x + y = 1$ $2x + 2y = 2$	**Solution 1:** $\boldsymbol{x = 0, y = 1}$ **Solution 2:** $\boldsymbol{x = 1, y = 0}$ **Solution 3:** $\boldsymbol{x = 2, y = -1}$ **...** Infinite possibilities!

Notes

Worked Example

419d: Solve simultaneous equations/systems of equations given in the form $ax + by = c$ and $x^2 + y^2 = d$

Solve the following simultaneous equations.

$$\begin{cases} x + 3y = 4 \\ x^2 + y^2 = 50 \end{cases}$$

Worked Example

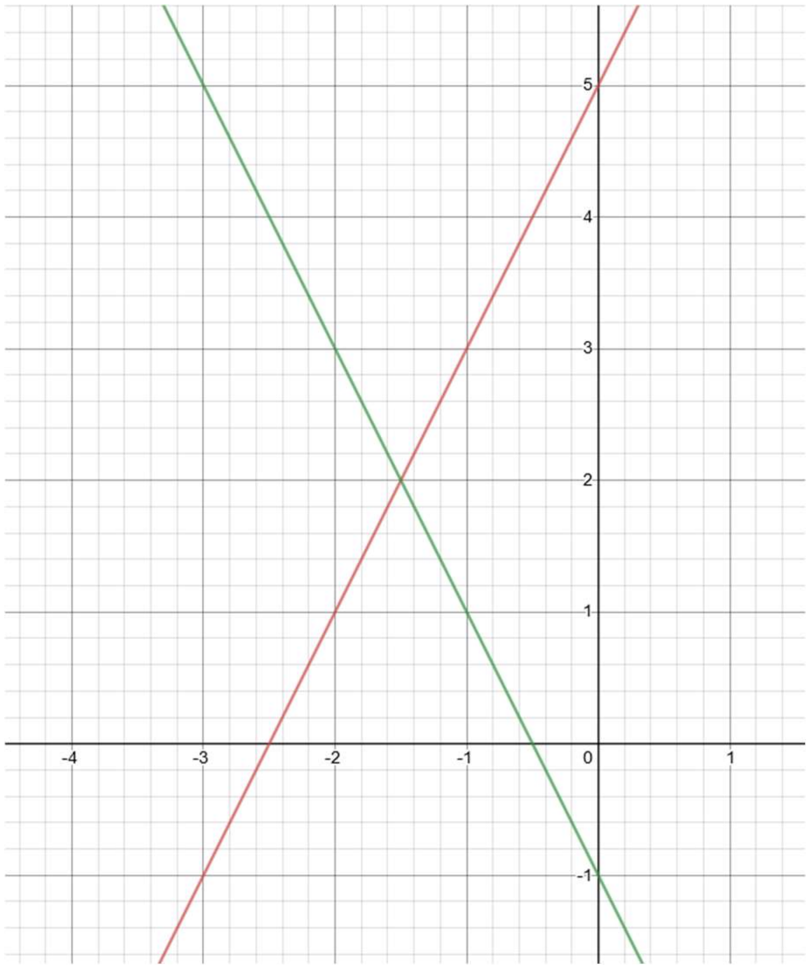
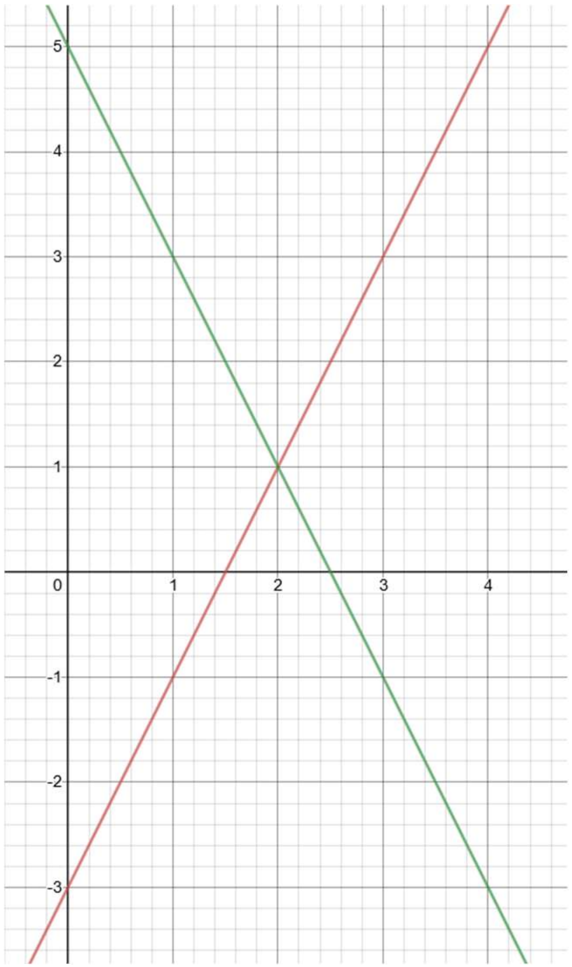
Solve:

$$xy = 12$$
$$x = y - 2$$

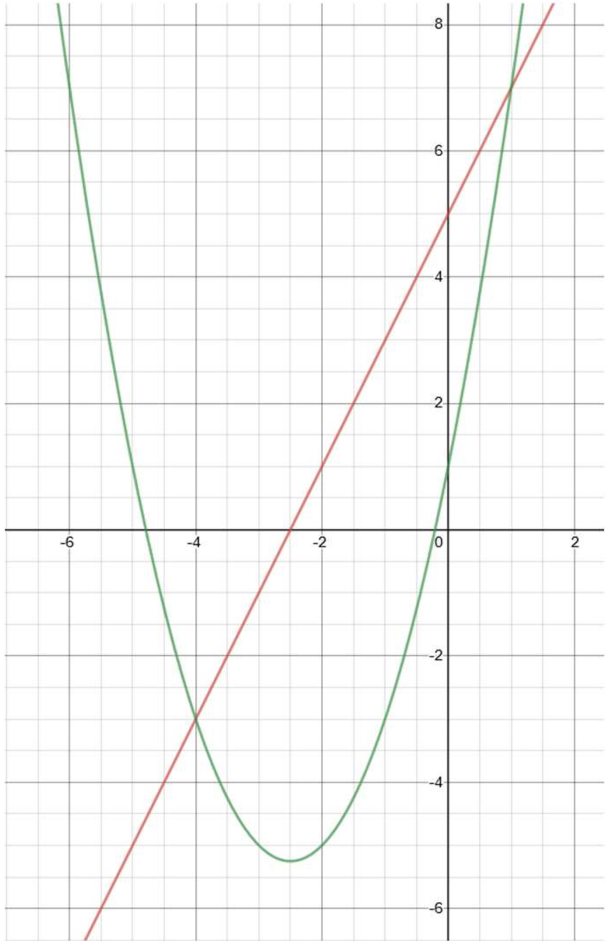
T.41 : 3B Qs all, P.17: 3.2 all Qs

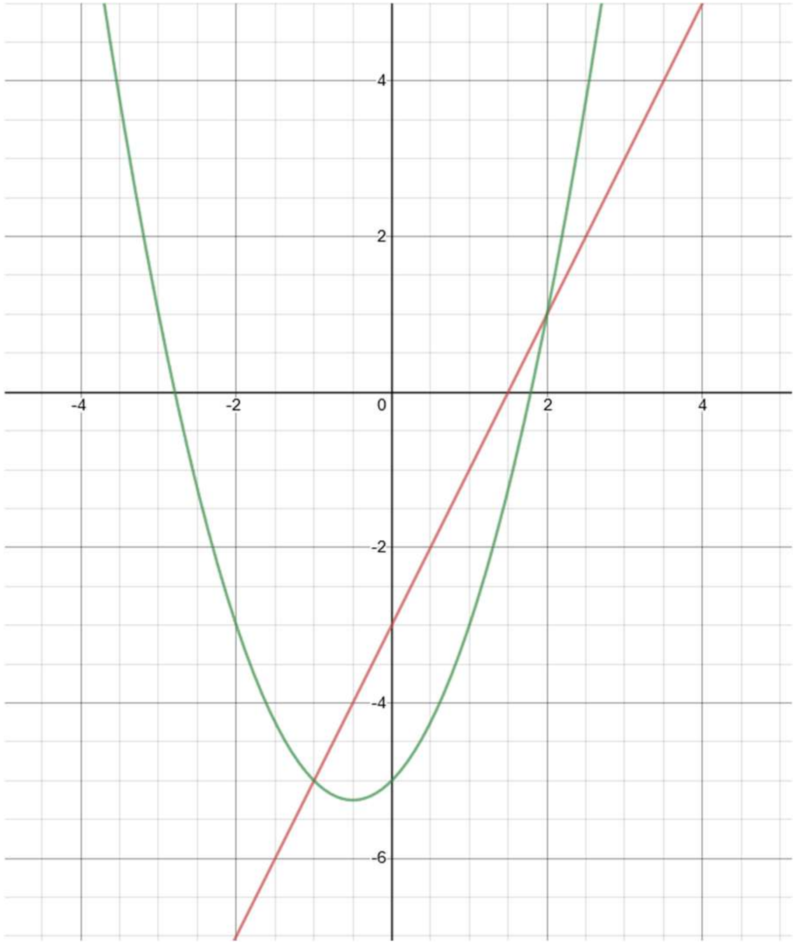
3.3) Simultaneous equations on graphs

Notes

Worked example	Your turn
Solve: $$y = 2x + 5$$ $$y = -2x - 1$$ 	Solve: $$y = 2x - 3$$ $$y = -2x + 5$$ 

Worked example	Your turn
Solve: $y = 2x + 5$, $y = x^2 + 5x + 1$	Solve: $y = 2x - 3$, $y = x^2 + x - 5$





T.45 : 3C Qs 1-4, P.18: 3.3 all Qs 1-4

Worked Example

The line with equation $y = 3x + 4$ meets the curve with equation $kx^2 + 2y + (k - 8) = 0$ at exactly one point. Given that k is a positive constant:

a) Find the value of k.

b) For this value of k, find the coordinates of this point of intersection.

T.45 : 3C Qs 8+, P.18: 3.3 all Qs 7+

3.4) Linear inequalities

Recall that a **set** is a **collection of values** such that:

a) The **order of values does not matter**.

b) There are **no duplicates**.

Recap from GCSE:

- We use curly braces to list the values in a set, e.g. $A = \{1,4,6,7\}$
- If A and B are sets then $A \cap B$ is the **intersection** of A and B, giving a set which has the elements in A **and** B.
- $A \cup B$ is the **union** of A and B, giving a set which has the elements in A **or** in B.
- $\emptyset$ is the empty set, i.e. the set with nothing in it.
- Sets can also be infinitely large. $\mathbb{N}$ is the set of natural numbers (all positive integers), $\mathbb{Z}$ is the set of all integers (including negative numbers and 0) and $\mathbb{R}$ is the set of all real numbers (including all possible decimals).
- We write $x \in A$ to mean "x is a member of the set A". So $x \in \mathbb{R}$ would mean "x is a real number".

$$\{1,2,3\} \cap \{3,4,5\} = \{\mathbf{3}\}$$
$$\{1,2,3\} \cup \{3,4,5\} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$$
$$\{1,2\} \cap \{3,4\} = \emptyset$$

Set Builder Notation

It is possible to construct sets without having to explicitly list its values. We use:

The | or : means "such that".

$$\{expr \mid condition\}$$

or

$$\{expr : condition\}$$

Can you guess what sets the following give?

(In words "All numbers $2x$ such that x is an integer)

$$\{2x : x \in \mathbb{Z}\} = \{0, 2, -2, 4, -4, 6, -6, \ldots\}$$

i.e. The set of all even numbers!

$$\{2^x : x \in \mathbb{N}\} = \{2, 4, 8, 16, 32, \ldots\}$$

$$\{xy : x, y \text{ are prime}\} = \{4, 6, 10, 14, 15, \ldots\}$$

i.e. All possible products of two primes.

We previously talked about 'solutions sets', so set builder notation is very useful for specifying the set of solutions!

Examples

All odd numbers.

$$\{2x + 1 \ : \ x \in \mathbb{Z}\}$$

All (real) numbers greater than 5.

$$\{x: \ x > 5\}$$

Technically it should be $\{x: \ x > 5, x \in \mathbb{R}\}$ but the $x > 5$ by default implies **real numbers** greater than 5.

All (real) numbers less than 5 **or** greater than 7.

$$\{x: \ x < 5\} \cup \{x: x > 7\}$$

We combine the two sets together.

All (real) numbers between 5 and 7 inclusive.

$$\{x: \ 5 \leq x \leq 7\}$$

While we could technically write $\{x: x \geq 5\} \cap \{x: x \leq 7\}$, we tend to write multiple required conditions within the same set.

Notes

Fill in the blanks

Inequality/Inequalities	In Set Notation
$x \geq 3.2$	$\{x: x \geq 3.2\}$
$-2 \leq x < 5$	
	$\{x: x < -8\} \cup \{x: x \geq 3\}$
$-\frac{1}{2} < x < 4$	$\{\quad\} \cap \{\quad\}$
	$\left\{x: -\frac{1}{2} < x < 4\right\}$
$x < \frac{7}{2}$ and $x > 1$ $x \leq 1$ or $x \geq 6.5$	
$-\frac{3}{4} \leq x \leq -\frac{1}{4}$	
	$\{x: x \leq -3\} \cup \{x: 2 < x < 5\}$
$x > 10$ or $-1 \leq x < 0$	
	$\{x: x \geq 5\} \cup \left\{x: x < \frac{3}{2}\right\}$
	$\left\{x: -2 \leq x < \frac{3}{5}\right\} \cup \{x: x > 1\}$
$x < 9$ and $x > \frac{2}{3}$	

Worked Example

Use set notation to describe the set of values for which:

$$10(9x+8)<7 \text{ or } 6(5x-4)\geq\frac{3-2x}{4}$$

T.48: 3D Q3 , P.19: 3.4 Q5-8

3.5) Quadratic inequalities

Recall you need to avoid multiplying (or dividing) by a negative. **This includes a variable x** *(as it could be negative).*

E.g. $\frac{a}{x} < b$ **becomes** $ax < bx^2$

You need to remember how to use dotted and solid line convention for inequalities, i.e.

$>$ *or* $<$ go with dotted (or hollow circle)
$\geq$ *or* $\leq$ with solid (or shaded circle)

Notes

Fill in the blanks

First Inequality	Solution to 1st Inequality	2nd Inequality	Solution to 2nd Inequality	Combined Solution
$2x - 1 \geq 4x + 3$		$3(7 + x) > 6$		$-5 < x \leq -2$
$\frac{3 + 2x}{4} < 3$		$5x - 2 < 5 + x$		
$3(x - 3) > 2x - 11$		$5 - x \leq 7 - 4x$		
$7(2x + 1) > 4x - 3$		$x^2 - 2x - 8 \leq 0$		
$x^2 - 7x + 10 < 0$		$6(1 - x) \leq 0$		
$\frac{5x + 1}{8} > 2$		$2x^2 - 9x \geq 5$		
$2(2 - 3x^2) \geq 5x$		$12 - 5x < 9 - 2x$		
$2x^2 < 50$		$5x(x - 6) \leq 8x - 21$		

Worked Example

Find the set of values for which $\frac{10}{x} > 5, x \neq 0$

Worked Example

Find the set of values for which $\frac{5}{x-3} < 2$

T.50: 3E Q 4 , P.20: 3.5 Q3

Worked Example

458m: Solve quadratic inequalities involving a division by a positive algebraic expression.

Solve

$$1 < \frac{7x}{2x^2 + 3}$$

Worked Example

4580: Combine a linear and a quadratic inequality.

n is an integer such that $7n-4 \geq 24$ and $\dfrac{5n}{n^2+4} \geq 1$

Find all the possible values of n.

Worked Example

The equation $kx^2 - 5kx + 50 = 0$, where k is a constant, has no real roots.
Prove that k satisfies the inequality $0 \leq k < 8$

T.50: 3E Qs4+, P.20: 3.5 Qs 4+

3.6) Inequalities on graphs

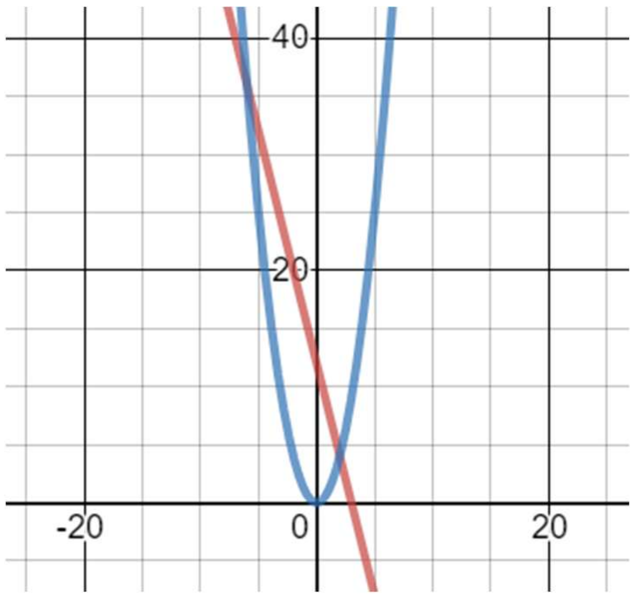
Notes

Worked Example

L_1 has equation $y = 12 - 4x$.
L_2 has equation $y = x^2$.
The diagram shows a sketch of L_1 and L_2 on the same axes.

a) Find the coordinates of the points of intersection.

b) Hence write down the solution to the inequality $12 - 4x > x^2$



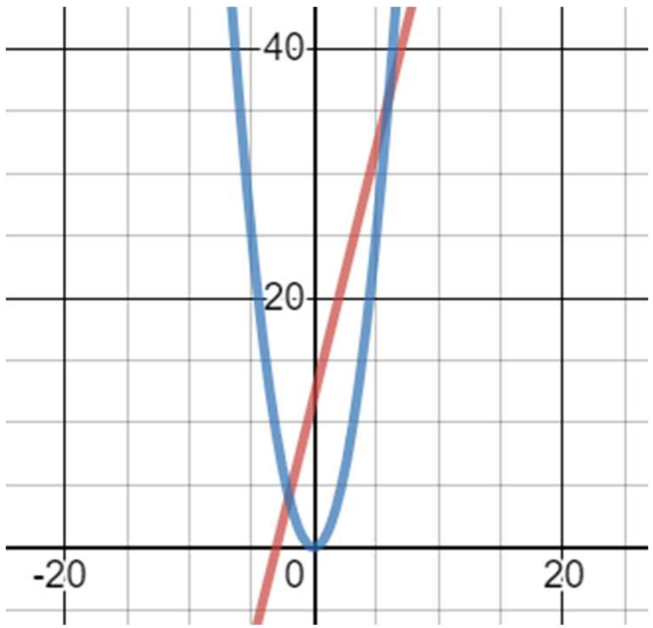
T.52: 3F all Q+, P.21: 3.6 all Qs

Your Turn

L_1 has equation $y = 12 + 4x$.
L_2 has equation $y = x^2$.
The diagram shows a sketch of L_1 and L_2 on the same axes.

a) Find the coordinates of the points of intersection.

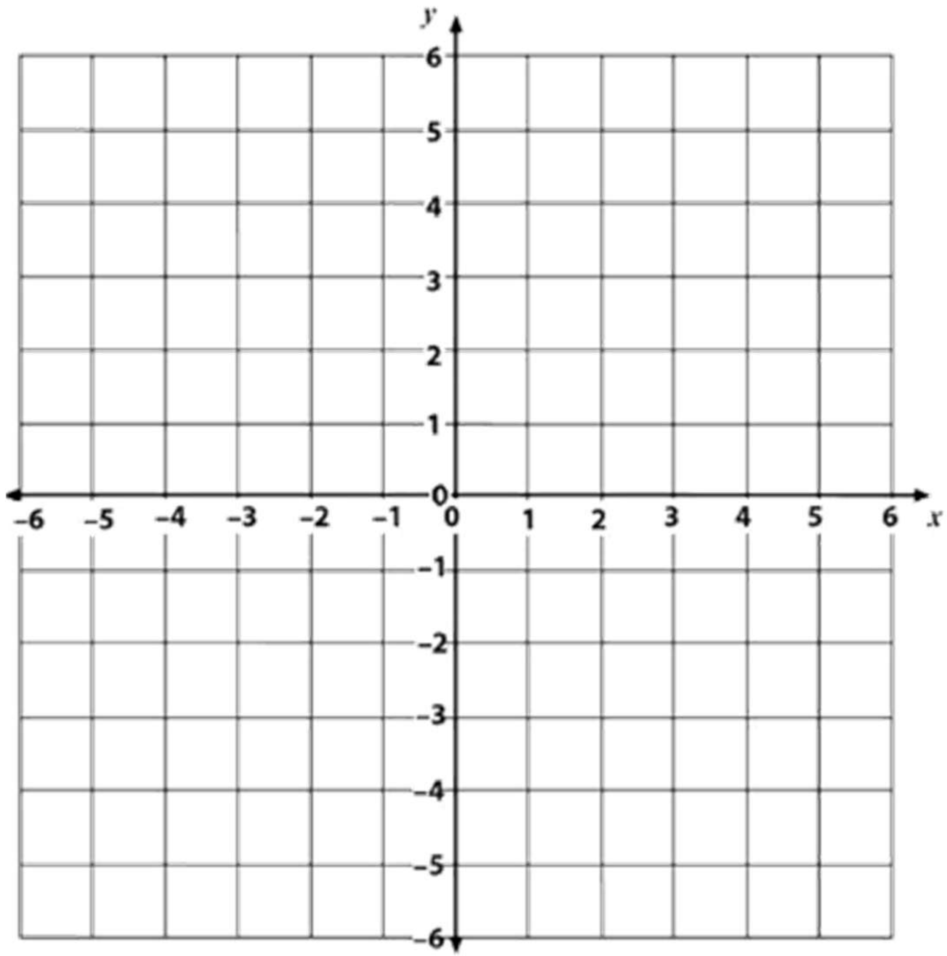
b) Hence write down the solution to the inequality $12 + 4x > x^2$

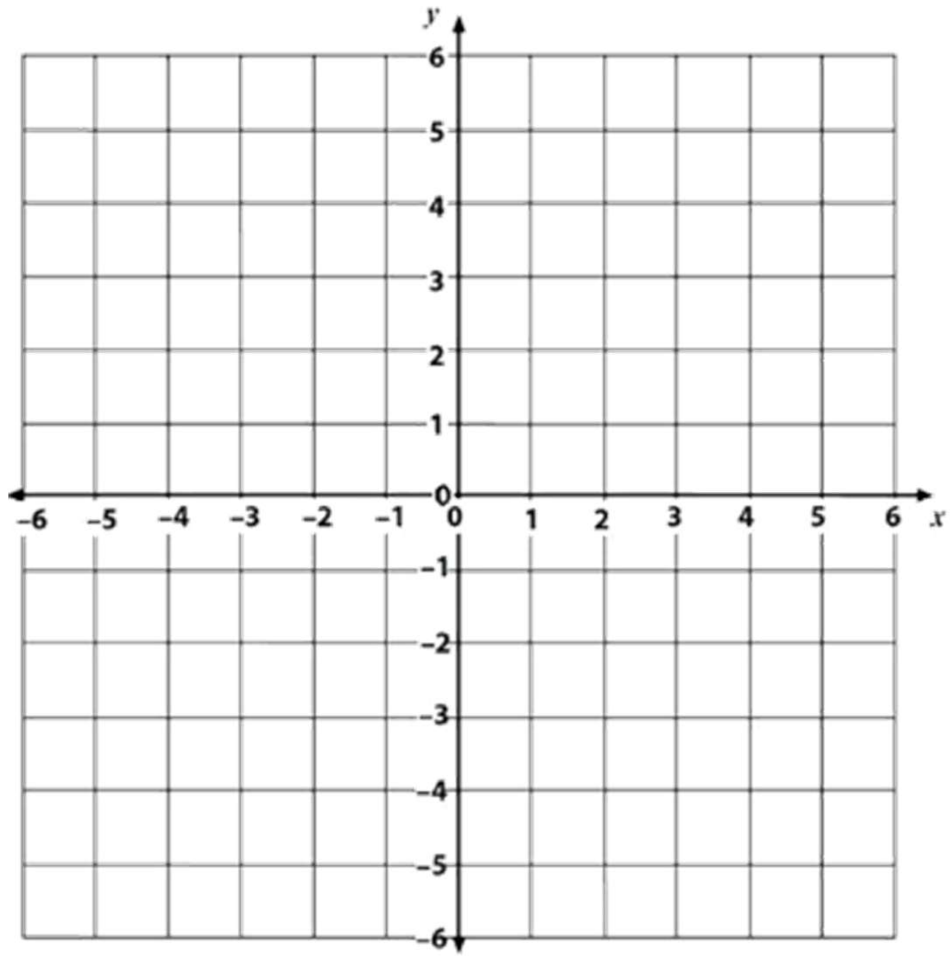


3.7) Regions

Notes

Worked example	Your turn
Shade the region which satisfies the inequalities: $x \geq -2, y < 1$ and $y < x - 1$	Shade the region which satisfies the inequalities. Label it R. $x > -3, y \leq 4$ and $y < x - 2$

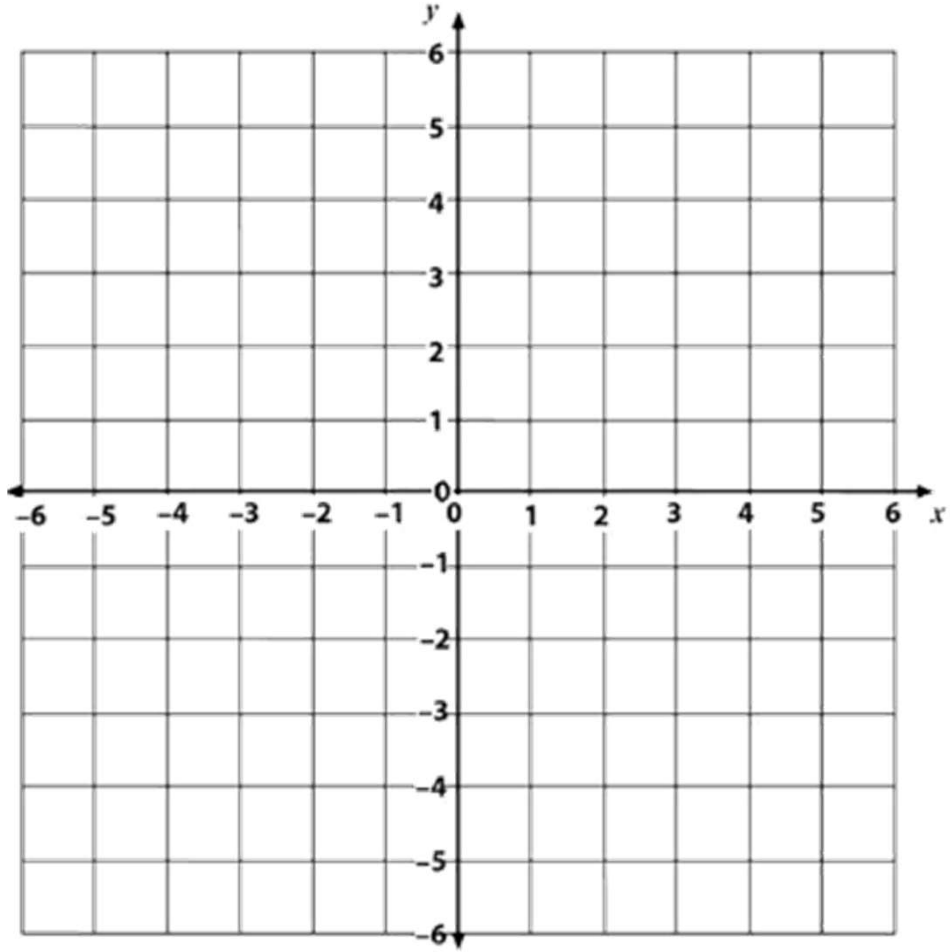




Worked example

Shade the region which satisfies the inequalities:

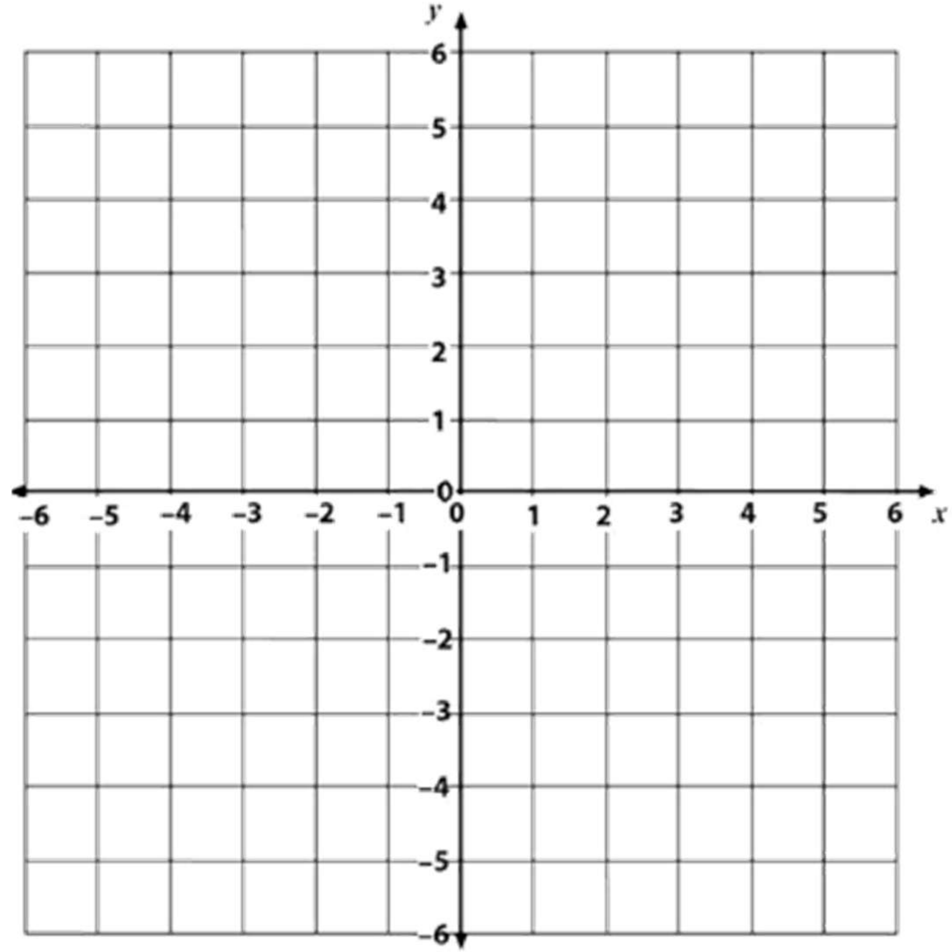
$x \geq 2, y > -1$ and $x + y \leq 5$



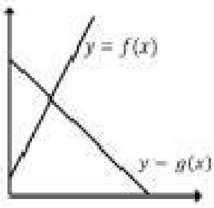
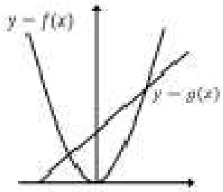
Your turn

Shade the region which satisfies the inequalities. Label it R.

$x \geq 2, y > 1$ and $x + y \leq 6$



Fill in the blanks

$f(x)$ and $g(x)$	Sketch of $y = f(x)$ and $y = g(x)$	Coordinates of intersection(s)	Solutions to $f(x) \geq g(x)$	Shade the region given by:
$f(x) = 3x + 1$ $g(x) = 5 - x$				$y \geq 3x + 1$ $y \leq 5 - x$ and $x \geq 0$
$f(x) = 2x - \frac{1}{2}$ $g(x) = \frac{1}{3}x + 2$				$y \geq 2x - \frac{1}{2}$ $y \leq \frac{1}{3}x + 2$ $x \geq 0$ and $y \geq 0$
$f(x) = x^2$ $g(x) = x + 6$				$y \geq x^2$ and $y \leq x + 6$
$f(x) = 1 - x^2$ $g(x) = x - 1$				$y \leq 1 - x^2$ and $y \leq x - 1$
$f(x) = 4 - 2x$ $g(x) = x^2 + x - 6$				$y \leq 4 - 2x$ and $y \leq x^2 + x - 6$
$f(x) = 4 - x^2$ $g(x) = 2x^2 + 4x$				$y \leq 4 - x^2$ and $y \geq 2x^2 + 4x$

Worked Example
Shade the region that satisfies the inequalities: $$4y + x \leq 12$$ $$y > x^2 - 5x - 6$$

1. **In this question you should show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Using algebra, solve the inequality

$$x^2 - x > 20$$

writing your answer in set notation.

(3)

Your Turn

1. **In this question you should show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

Using algebra, solve the inequality

$$x^2 - x > 30$$

writing your answer in set notation.

(3)

(Total for Question 1 is 3 marks)

7.

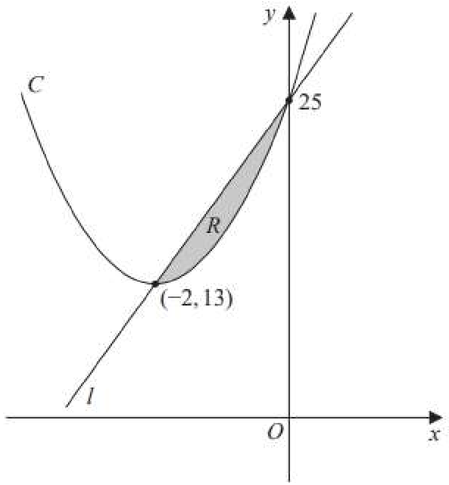


Figure 1

Figure 1 shows a sketch of a curve C with equation $y = f(x)$ and a straight line l.

The curve C meets l at the points $(-2, 13)$ and $(0, 25)$ as shown.

The shaded region R is bounded by C and l as shown in Figure 1.

Given that

- $f(x)$ is a quadratic function in x
- $(-2, 13)$ is the minimum turning point of $y = f(x)$

use inequalities to define R.

(5)

Your Turn

7.

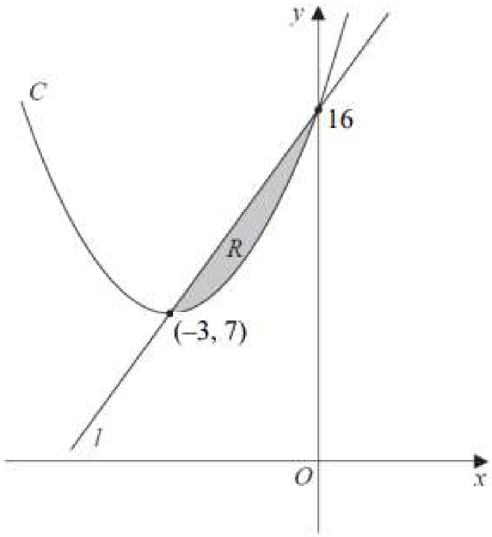


Figure 1

Figure 1 shows a sketch of a curve C with equation $y = f(x)$ and a straight line l.

The curve C meets l at the points $(-3, 7)$ and $(0, 16)$ as shown.

The shaded region R is bounded by C and l as shown in Figure 1.

Given that $f(x)$ is a quadratic function in x and that $(-3, 7)$ is the minimum turning point of $y = f(x)$, use inequalities to define R.

(Total for Question 7 is 5 marks)

1. **In this question you should show all stages of your working.**

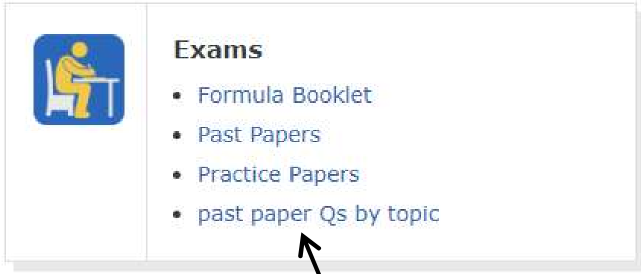
Solutions relying on calculator technology are not acceptable.

Using algebra, solve the inequality

$$x^2 - x > 20$$

writing your answer in set notation.

(3)



Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Question	Scheme	Marks	AOs
1	Finds critical values $x^2 - x > 20 \Rightarrow x^2 - x - 20 > 0 \Rightarrow x = (5, -4)$	M1	1.1b
	Chooses outside region for their values Eg. $x > 5, x < -4$	M1	1.1b
	Presents solution in set notation $\{x : x < -4\} \cup \{x : x > 5\}$ oe	A1	2.5
		(3)	
(3 marks)			

Summary of Key Points

Summary of key points

1. Linear simultaneous equations can be solved using elimination or substitution.
2. Simultaneous equations with one linear and one quadratic equation can have up to two pairs of solutions. You need to make sure the solutions are paired correctly.
3. The solutions of a pair of simultaneous equations represent the points of intersection of their graphs.
4. For a pair of simultaneous equations that produce a quadratic equation of the form $ax^2 + bx + c = 0$:
   - $b^2 - 4ac > 0$ two real solutions
   - $b^2 - 4ac = 0$ one real solution
   - $b^2 - 4ac < 0$ no real solutions
5. The solution of an inequality is the set of all real numbers x that make the inequality true.
6. To solve a quadratic inequality:
   - Rearrange so that the right-hand side of the inequality is 0
   - Solve the corresponding quadratic equation to find the critical values
   - Sketch the graph of the quadratic function
   - Use your sketch to find the required set of values.
7. The values of x for which the curve $y = f(x)$ is **below** the curve $y = g(x)$ satisfy the inequality $f(x) < g(x)$.
   The values of x for which the curve $y = f(x)$ is **above** the curve $y = g(x)$ satisfy the inequality $f(x) > g(x)$.
8. $y < f(x)$ represents the points on the coordinate grid below the curve $y = f(x)$.
   $y > f(x)$ represents the points on the coordinate grid above the curve $y = f(x)$.
9. If $y > f(x)$ or $y < f(x)$ then the curve $y = f(x)$ is not included in the region and is represented by a dotted line.
   If $y \geqslant f(x)$ or $y \leqslant f(x)$ then the curve $y = f(x)$ is included in the region and is represented by a solid line.