KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Pure Mathematics
Chapter 14 pt 2 Exponentials and Logarithms

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

All key skills 527 :

[Chp14 - Exponentials & Logarithms (drfrost.org)]

527 Laws of logs (excluding $\ln(x)$)

Mastery: 0/100

Set a Task | Generate Worksheet | Have a Go

OR NARROW DOWN		VIDEO	DIFFICULTY
☐ 527: Exam Practice: Laws of logs (excluding $\ln(x)$)	Browse		1-4
☐ 527b: Convert from index form to logarithmic form.	Example		1
☐ 527c: Solve logarithmic equations given in the form $\log_a x = b$	Example		2
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14.1) Exponential functions

Functions of the form $f(x) = a^x$, where a is a constant, are called **exponential functions**. You should become familiar with these functions and the shapes of their graphs.

For an example, look at a table of values of $y = 2^x$.

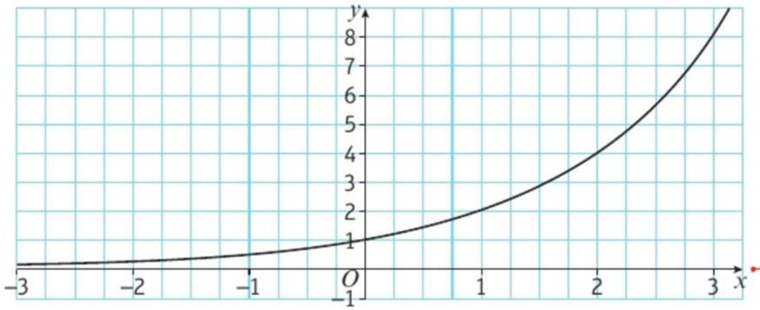
x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Notation In the expression 2^x, x can be called an **index**, a **power** or an **exponent.**

Links Recall that $2^0 = 1$ and that $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ ← Section 1.4

The value of 2^x tends towards 0 as x decreases, and grows without limit as x increases.

The graph of $y = 2^x$ is a smooth curve that looks like this:



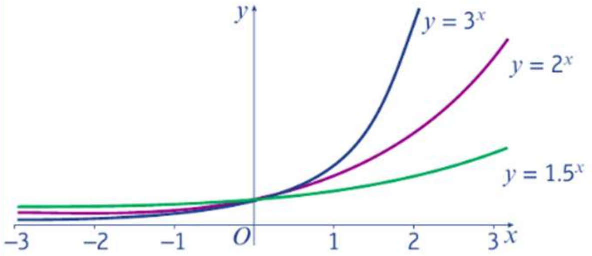
The x-axis is an asymptote to the curve.

Notes

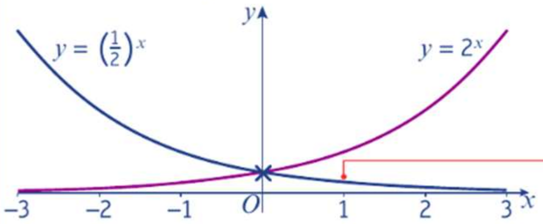
a For all three graphs, $y = 1$ when $x = 0$.

When $x > 0$, $3^x > 2^x > 1.5^x$.

When $x < 0$, $3^x < 2^x < 1.5^x$.



b The graph of $y = \left(\frac{1}{2}\right)^x$ is a reflection in the y-axis of the graph of $y = 2^x$.



$a^0 = 1$

Work out the relative positions of the three graphs.

Whenever $a > 1$, $f(x) = a^x$ is an increasing function. In this case, the value of a^x grows without limit as x **increases**, and tends towards 0 as x **decreases**.

Since $\frac{1}{2} = 2^{-1}$, $y = \left(\frac{1}{2}\right)^x$ is the same as $y = (2^{-1})^x = 2^{-x}$.

Whenever $0 < a < 1$, $f(x) = a^x$ is a decreasing function. In this case, the value of a^x tends towards 0 as x **increases**, and grows without limit as x **decreases**.

Your Turn

On the same axes, sketch $y = 4^x$, $y = 5^x$ and $y = 3.5^x$

Your Turn

On the same axes, sketch $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$

Worked Example

Sketch $y = 3^{x-2}$

T.313: 14A Qs4-5, P.113: 14.1 Q4-6

Worked Example

The graph of $y = ka^x$ passes through the points $(4, \frac{16}{3})$ and $(0, \frac{1}{3})$

Find the values of the constants k and a

T.313: 14A Qs6-7, P.113: 14.1 Q7

14.2) $y = e^x$

- **For all real values of x:**
  - **If $f(x) = e^x$ then $f'(x) = e^x$**
  - **If $y = e^x$ then $\frac{dy}{dx} = e^x$**

A similar result holds for functions such as e^{5x}, e^{-x} and $e^{\frac{1}{2}x}$.

- **For all real values of x and for any constant k:**
  - **If $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$**
  - **If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$**

Notes

Worked Example

Differentiate with respect to x:

$a)$ e^{2x}

$b)$ e^{-3}

c) $4e^{5x}$

$d)$ $6e^{\frac{1}{3}x}$

T.316: 14B Qs3-5,7 ,P.114: 14.2 Qs 2, 3, 5,6,8

Worked Example

Sketch

$$y = e^{2x}\ , y = e^{3x}, y = e^{-2x}$$

On the same axis as $y = e^{x}$

Worked Example

Sketch:

$$y = 3 + 4e^{\frac{1}{2}x}$$

Worked Example

Sketch:

$$y = e^{-3x} - 2$$

T.316: 14B Qs3-5, P.114: 14.3 Qs 1, 4,7

14.3) Exponential modelling

Notes

Worked Example

Suppose the population P of a village is modelled by $P = 500e^{2t}$ where t is the numbers of years since February 2009. Find:

a) The initial population

b) The initial rate of growth

c) The population in February 2014

T.318: 14C Qs1-3

Worked Example

The density of a pesticide in a given section of field, P mg/m², can be modelled by the equation $P = 80e^{-0.003t}$ where t is the time in days since the pesticide was first applied.

a. Use this model to estimate the density of pesticide after 30 days.

b. Interpret the meaning of the value 80 in this model.

c. Show that $\frac{dP}{dt} = kP$, where k is a constant, and state the value of k.

d. Interpret the significance of the sign of your answer in part (c).

e. Sketch the graph of P against t.

T.318: 14B Qs3+, P.115: 14.3 all Qs

4. Coffee is poured into a cup.

The temperature of the coffee, H °C, t minutes after being poured into the cup is modelled by the equation

$$H = Ae^{-Bt} + 30$$

where A and B are constants.

Initially, the temperature of the coffee was 85 °C.

(a) State the value of A.

(1)

Initially, the coffee was cooling at a rate of 7.5 °C per minute.

(b) Find a complete equation linking H and t, giving the value of B to 3 decimal places.

(3)

Your Turn

4. Coffee is poured into a cup.

The temperature of the coffee, H °C, t minutes after being poured into the cup is modelled by the equation

$$H = Ae^{-Bt} + 25$$

where A and B are constants.

Initially, the temperature of the coffee was 90 °C.

(a) State the value of A.

(1)

Initially, the coffee was cooling at a rate of 8 °C per minute.

(b) Find a complete equation linking H and t, giving the value of B to 3 decimal places.

(3)

(Total for Question 4 is 4 marks)

14.7) Working with natural logarithms

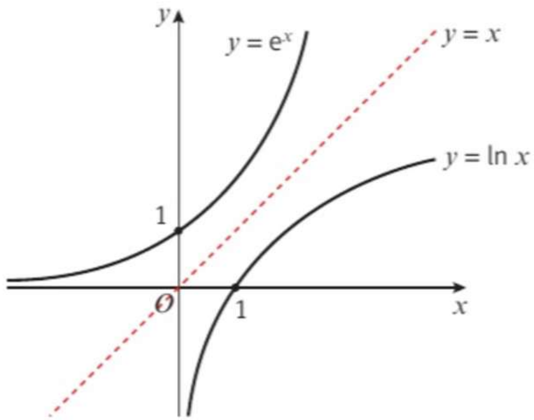
- **The graph of $y = \ln x$ is a reflection of the graph $y = e^x$ in the line $y = x$.**

The graph of $y = \ln x$ passes through (1, 0) and does not cross the y-axis.

The y-axis is an asymptote of the graph $y = \ln x$. This means that $\ln x$ is only defined for positive values of x.

As x increases, $\ln x$ grows without limit, but relatively slowly.

You can also use the fact that logarithms are the inverses of exponential functions to solve equations involving powers and logarithms.



- **$e^{\ln x} = \ln (e^x) = x$**

Notation $\ln x = \log_e x$

Notes

Worked Example

528b: **Solve exponential equations involving a single occurrence of e**

Solve for x:

$$3e^{3x+1} = 7$$

Give your solution in exact form.

Worked Example

Solve the equation:

$$3\ln x - 7 = 5$$

T.327: 14G Qs 1,2, P.119: 14.7 Qs 1,2

Worked Example

528g: **Solve equations which are quadratic in e^x**

Find the exact solution of

$$3e^{2x} + e^x = 10.$$

Input note: Give any answers involving logs in the form $\log_a b$.

Worked Example

Solve the equation:

$$e^x - 12e^{-x} = -1$$

Worked Example

528c: **Solve logarithmic equations involving a single occurrence of $\ln$**

Solve for x:

$$3\ln(-3x-3)+3=9$$

Give your solution in exact form.

Worked Example

Find the exact coordinates of the points where the graph with equation $y = 6 + \ln(5 - x)$ intersects the axes

T.327: 14G Qs 3,4, P.119: 14.7 Qs 3-6

Worked Example

Solve the equation:

$$3^x e^{x+4} = 2$$

Give your answer as an exact value

T.327: 14G Qs 5+ P.119: 14.7 Qs 7+

14.8) Logarithms and non-linear data

Case 1: $y = ax^n$

Start with a non-linear relationship	$y = ax^n$
Take logs of both sides ($\log = \log_{10}$)	$\log y = \log ax^n$
Use the multiplication law	$\log y = \log a + \log x^n$
Use the power law	$\log y = \log a + n \log x$

Compare this equation to the straight line equation, $Y = MX + C$.

$\log y$ variable	=	n constant (gradient)	$\log x$ variable	+	$\log a$ constant (intercept)
Y variable	=	M constant (gradient)	X variable	+	C constant (intercept)

- **If $y = ax^n$ then the graph of $\log y$ against $\log x$ will be a straight line with gradient n and vertical intercept $\log a$.**

Notes

Case 2: $y = ab^x$

Start with a non-linear relationship	$y = ab^x$
Take logs of both sides ($\log = \log_{10}$)	$\log y = \log ab^x$
Use the multiplication law	$\log y = \log a + \log b^x$
Use the power law	$\log y = \log a + x \log b$

Compare this equation to the straight line equation, $Y = MX + C$.

$\log y$		$\log b$	x		$\log a$
variable	=	constant (gradient)	variable	+	constant (intercept)
Y		M	X		C
variable	=	constant (gradient)	variable	+	constant (intercept)

Notes

Worked Example

Use logarithms to convert the non-linear relationship into a linear form and sketch the resulting straight line.

$$y = ax^n$$

T.331: 14H Qs 1,2 P.120: 14.8 Qs 1,2

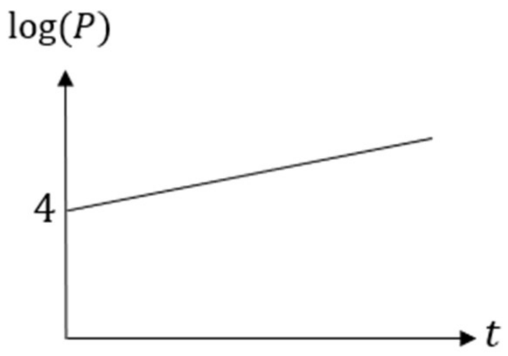
Worked Example

The graph represents the growth of a population of bacteria, P, over t hours.
The graph has a gradient of 0.3 and meets the vertical axis at $(0,4)$.
A scientist suggest that this growth can be modelled by the equation $P = ab^t$, where a and b are constants to be found.

a. Write down an equation for the line.

b. Find the values of a and b, giving them to 3 sf where necessary.

c. Interpret the meaning of the constant a in this model.



Worked Example

The table below gives the rank (by size) and population of a country's largest cities and districts (the capital city is number 1 but has been excluded as an outlier).

City	A	B	C	D	E
Rank, R	2	3	4	5	6
Population	2 000 000	1 400 000	1 200 000	1 000 000	900 000

The relationship between the rank and population can be modelled by the formula:

$P = aR^n$ where a and n are constants.

a) Draw a table giving values of $\log R$ and $\log P$ to 2dp.

b) Plot a graph of $\log R$ against $\log P$ using the values from your table and draw the line of best fit.

c) Use your graph to estimate the values of a and n to two significant figures.

Worked Example

A population is increasing exponentially according to the model $P = ab^t$, where a, b are constants to be found.

The population is recorded as follows:

Years t after 2016	1.4	2.6	4.4
Population P	4706	7346	14324

a) Draw a table giving values of t and $\log P$ (to 3dp).

b) A line of best fit is drawn for the data in your new table, and it happens to go through the first data point above (where $t = 1.4$) and last (where $t = 4.4$).
Determine the equation of this line of best fit.

c) Hence, determine the values of a and b in the model.

d) Estimate the population in 2020

T.331: 14H Qs 3+ P.120: 14.8 Qs 3+

8. The temperature, θ°C, of a cup of tea t minutes after it was placed on a table in a room, is modelled by the equation

$$\theta = 18 + 65e^{-\frac{t}{8}} \qquad t \geqslant 0$$

Find, according to the model,

(a) the temperature of the cup of tea when it was placed on the table,
(1)

(b) the value of t, to one decimal place, when the temperature of the cup of tea was 35 °C.
(3)

(c) Explain why, according to this model, the temperature of the cup of tea could not fall to 15 °C.
(1)

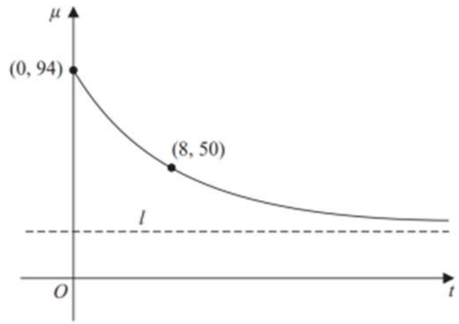


Figure 2

The temperature, μ°C, of a second cup of tea t minutes after it was placed on a table in a different room, is modelled by the equation

$$\mu = A + Be^{-\frac{t}{8}} \qquad t \geqslant 0$$

where A and B are constants.

Figure 2 shows a sketch of μ against t with two data points that lie on the curve.

The line l, also shown on Figure 2, is the asymptote to the curve.

Using the equation of this model and the information given in Figure 2

(d) find an equation for the asymptote l.

Your Turn

8 The temperature, θ °C, of a cup of tea t minutes after it was placed on a table in a room, is modelled by the equation

$$\theta = 16 + 69e^{-\frac{t}{8}} \qquad t \geqslant 0$$

Find, according to the model,

(*a*) the temperature of the cup of tea when it was placed on the table,

(1)

(*b*) the value of t, to two decimal places, when the temperature of the cup of tea was 40 °C.

(3)

(*c*) Explain why, according to this model, the temperature of the cup of tea could not fall to 10 °C.

(1)

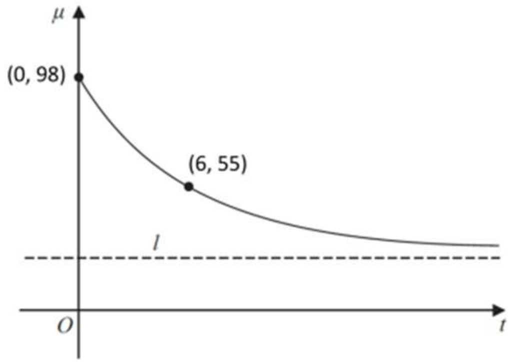


Figure 2

The temperature, μ °C, of a second cup of tea t minutes after it was placed on a table in a different room, is modelled by the equation

$$\mu = A + Be^{-\frac{t}{8}} \qquad t \geqslant 0$$

where A and B are constants.

Figure 2 shows a sketch of μ against t with two data points that lie on the curve.

The line l, also shown on Figure 2, is the asymptote to the curve.

Using the equation of this model and the information given in Figure 2

(*d*) find an equation for the asymptote l.

(4)

(Total for Question 8 is 9 marks)

12. An advertising agency is monitoring the number of views of an online advert.

The equation

$$\log_{10} V = 0.072t + 2.379 \qquad 1 \leqslant t \leqslant 30,\ t \in \mathbb{N}$$

is used to model the total number of views of the advert, V, in the first t days after the advert went live.

(a) Show that $V = ab^t$ where a and b are constants to be found.

Give the value of a to the nearest whole number and give the value of b to 3 significant figures.

(

(b) Interpret, with reference to the model, the value of ab.

(

Using this model, calculate

(c) the total number of views of the advert in the first 20 days after the advert went live. Give your answer to 2 significant figures.

(

Your Turn

12 An advertising agency is monitoring the number of views of an online advert.

The equation

$$\log_8 V = 0.068t + 2.589 \qquad 1 \leqslant t \leqslant 30,\ t \in \mathbb{N}$$

is used to model the total number of views of the advert, V, in the first t days after the advert went live.

(*a*) Show that $V = ab^t$ where a and b are constants to be found.

Give the value of a to the nearest whole number and give the value of b to 3 significant figures.

(4)

(*b*) Interpret, with reference to the model, the value of ab.

(1)

Using this model, calculate

(*c*) the total number of views of the advert in the first 15 days after the advert went live. Give your answer to 2 significant figures.

(2)

(Total for Question 12 is 7 marks)

5. The mass, A kg, of algae in a small pond, is modelled by the equation

$$A = pq^t$$

where p and q are constants and t is the number of weeks after the mass of algae was first recorded.

Data recorded indicates that there is a linear relationship between t and $\log_{10} A$ given by the equation

$$\log_{10} A = 0.03t + 0.5$$

(a) Use this relationship to find a complete equation for the model in the form

$$A = pq^t$$

giving the value of p and the value of q each to 4 significant figures.

(4)

(b) With reference to the model, interpret

(i) the value of the constant p,

(ii) the value of the constant q.

(2)

(c) Find, according to the model,

(i) the mass of algae in the pond when $t = 8$, giving your answer to the nearest 0.5 kg,

(ii) the number of weeks it takes for the mass of algae in the pond to reach 4 kg.

(3)

(d) State one reason why this may not be a realistic model in the long term.

(1)

Your Turn

5. The mass, M grams of a colony of bacteria, is modelled by the equation

$$M = pq^t$$

where p and q are constants and t is the number of days after the mass was first recorded.

Data recorded indicates that there is a linear relationship between t and $\log_{10} M$ given by the equation

$$\log_{10} M = 0.02t + 0.6$$

(*a*) Use this relationship to find a complete equation for the model in the form

$$M = pq^t$$

giving the value of p and the value of q each to 4 significant figures.

(4)

(*b*) With reference to the model, interpret

(i) the value of the constant p,

(ii) the value of the constant q.

(2)

(*c*) Find, according to the model,

(i) the mass of the colony when $t = 5$, giving your answer to the nearest gram,

(ii) the number of days it takes for the mass of the colony to reach 7 grams.

(3)

(*d*) State one reason why this may not be a realistic model in the long term.

(1)

(Total for Question 5 is 10 marks)

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The air pressure, P kg/cm^2, inside a car tyre, t minutes from the instant when the tyre developed a puncture is given by the equation

$$P = k + 1.4e^{-0.5t} \qquad t \in \mathbb{R} \qquad t \geqslant 0$$

where k is a constant.

Given that the initial air pressure inside the tyre was 2.2 kg/cm^2

(a) state the value of k.

(1)

From the instant when the tyre developed the puncture,

(b) find the time taken for the air pressure to fall to 1 kg/cm^2
Give your answer in minutes to one decimal place.

(3)

(c) Find the rate at which the air pressure in the tyre is decreasing exactly 2 minutes from the instant when the tyre developed the puncture.
Give your answer in kg/cm^2 per minute to 3 significant figures.

Your Turn

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The volume of water, V litres, in a paddling pool, t minutes from the instant when it was punctured is given by the equation

$$V = k + 1900e^{-0.1t} \quad t \in \mathbb{R} \qquad t \geq 0$$

where k is a constant.

Given that the initial volume of water in the paddling pool was 2000 litres,

(*a*) state the value of k.

(1)

From the instant when the pool was punctured,

(*b*) find the time taken for the volume to fall to 500 litres
Give your answer in minutes to one decimal place.

(3)

(*c*) Find the rate at which the volume of water in the pool is decreasing exactly 20 minutes after the instant when it was punctured.
Give your answer in litres per minute to 3 significant figures.

(2)

(Total for Question 8 is 6 marks)

Extract from Formulae book

Logarithms and exponentials

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$e^{x \ln a} = a^x$$

A2 2019 Paper 1

Exponentials and Logs

7. In a simple model, the value, £V, of a car depends on its age, t, in years.

The following information is available for car A

- its value when new is £20 000
- its value after one year is £16 000

(a) Use an exponential model to form, for car A, a possible equation linking V with t. **(4)**

The value of car A is monitored over a 10-year period.
Its value after 10 years is £2 000

(b) Evaluate the reliability of your model in light of this information. **(2)**

The following information is available for car B

- it has the same value, when new, as car A
- its value depreciates more slowly than that of car A

(c) Explain how you would adapt the equation found in (a) so that it could be used to model the value of car B. **(1)**

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$V = Ae^{-kt}$	M1	This mark is given for suggesting a suitable exponential model for V in terms of t
	When $t = 0$ and $V = 20\,000$, $A = 20\,000$	M1	This mark is given for using the model to show the initial value for A is £20 000
	When $t = 1$ and $V = 16\,000$, $16\,000 = 20\,000e^{-1k}$ $k = \ln 0.8 = -0.223$	M1	This mark is given for using the value of the car after one year to find a value for k
	$V = 20\,000e^{-0.223t}$	A1	This mark is given for finding a fully correct exponential model
(b)	When $t = 10$, $V = £2150$	M1	This mark is given for finding a value for V when $t = 10$
	This model is reliable since the value £2150 is close to £2000	A1	This mark is given for a valid statement comparing the two possible values of the car after 10 years
(c)	For example: The value of k should be increased (e.g. $V = 20\,000e^{-0.1t}$) A constant should be added (e.g. $V = 20\,000e^{-0.223t} + 2000$)	B1	This mark is given for a statement suggesting a valid adaptation

Summary of Key Points

Summary of key points

1 For all real values of x:

- If $f(x) = e^x$ then $f'(x) = e^x$
- If $y = e^x$ then $\frac{dy}{dx} = e^x$

2 For all real values of x and for any constant k:

- If $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$
- If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$

3 $\log_a n = x$ is equivalent to $a^x = n$ $(a \neq 1)$

4 **The laws of logarithms:**

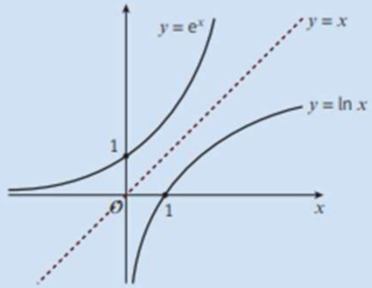
- $\log_a x + \log_a y = \log_a xy$ (the multiplication law)
- $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$ (the division law)
- $\log_a (x^k) = k \log_a x$ (the power law)

5 You should also learn to recognise the following special cases:

- $\log_a \left(\frac{1}{x}\right) = \log_a (x^{-1}) = -\log_a x$ (the power law when $k = -1$)
- $\log_a a = 1$ $(a > 0, a \neq 1)$
- $\log_a 1 = 0$ $(a > 0, a \neq 1)$

6 Whenever $f(x) = g(x)$, $\log_a f(x) = \log_a g(x)$

7 The graph of $y = \ln x$ is a reflection of the graph $y = e^x$ in the line $y = x$.



8 $e^{\ln x} = \ln (e^x) = x$

9 If $y = ax^n$ then the graph of $\log y$ against $\log x$ will be a straight line with gradient n and vertical intercept $\log a$.

10 If $y = ab^x$ then the graph of $\log y$ against x will be a straight line with gradient $\log b$ and vertical intercept $\log a$.

T.336 mixed ex. P.122: BSG