KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Pure Mathematics
P1 4 Graphs and Transformations
Booklet

HGS Maths

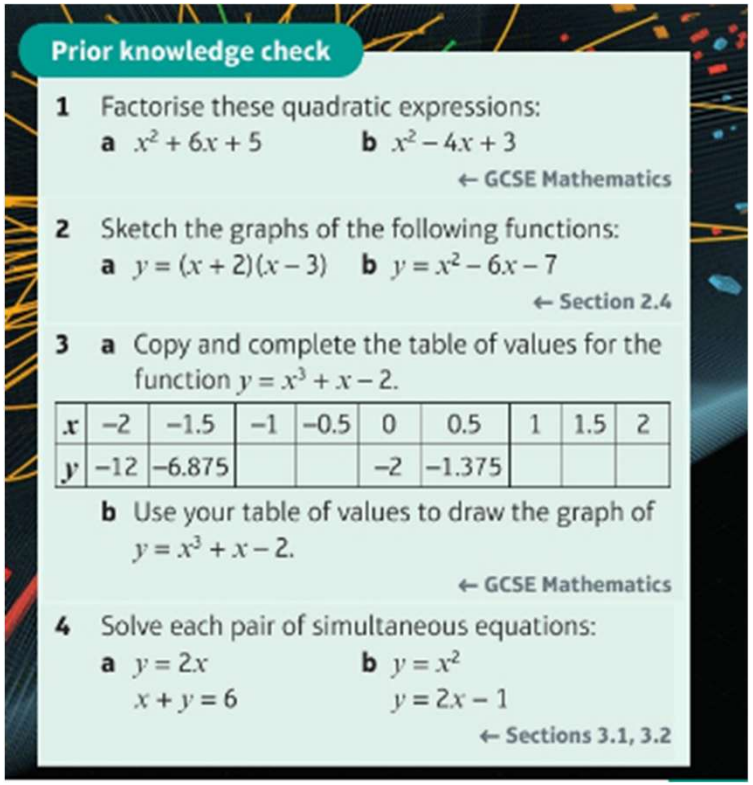
Dr Frost Course

Name: _____

Class: _____

Contents

Prior knowledge check



Prior knowledge check

1 Factorise these quadratic expressions:

a $x^2 + 6x + 5$ **b** $x^2 - 4x + 3$

← GCSE Mathematics

2 Sketch the graphs of the following functions:

a $y = (x + 2)(x - 3)$ **b** $y = x^2 - 6x - 7$

← Section 2.4

3 **a** Copy and complete the table of values for the function $y = x^3 + x - 2$.

x	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
y	−12	−6.875			−2	−1.375			

b Use your table of values to draw the graph of $y = x^3 + x - 2$.

← GCSE Mathematics

4 Solve each pair of simultaneous equations:

a $y = 2x$
$x + y = 6$

b $y = x^2$
$y = 2x - 1$

← Sections 3.1, 3.2

4.1) Cubic graphs

In Chapter 2 we briefly saw that a **polynomial** expression is of the form:

$$a + bx + cx^2 + dx^3 + ex^3 + \cdots$$

where $a, b, c, d, e, \ldots$ are constants (which could be 0).

The **order** of a polynomial is its highest power.

Order	Name
0	Constant (e.g. "4")
1	Linear (e.g. "$2x - 1$")
2	Quadratic (e.g. "$x^2 + 3$")
3	Cubic
4	Quartic
5	Quintic

These are covered in Chapter 5.

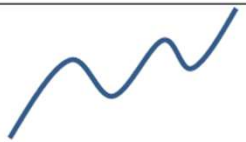
Chapter 2 explored the graphs for these.

We will cover these now.

While these are technically beyond the A Level syllabus, we will look at how to sketch polynomials in general.

Summary

e.g. If $y = 2x^2 + 3$, try a large positive value like $x = 1000$. We can see we'd get a large positive y value. Thus as $x \to \infty, y \to \infty$

Equation	If $a > 0$	Resulting Shape	If $a < 0$	Resulting Shape
$y = ax^2 + bx + c$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to \infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to -\infty$	
$y = ax^3 + bx^2 + cx + d$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to -\infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to \infty$	
$y = ax^4 + bx^3 + cx^2 + dx + e$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to \infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to -\infty$	
$y = ax^5 + bx^4 + \cdots$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to -\infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to \infty$	

If $a > 0$, what therefore can we say about the shape if:

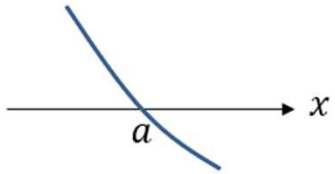
- **The order is odd**: It goes uphill (from left to right)
- **The order is even**: The tails go upwards.

(And we have the opposite if $a < 0$)

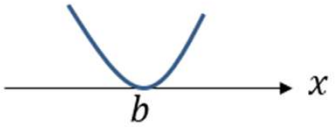
Important notes

If we sketched $y = (x - a)(x - b)^2(x - c)^3$ what happens on the x-axis at:

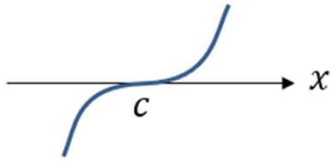
$x = a$: The line **crosses** the axis.



$x = b$: The line **touches** the axis.



$x = c$: **Point of inflection** on the axis.



Notes

Worked Example

Sketch the graph of:

$$y = x^3 - 4x^2 - 5x$$

Sketch the graph of:

$$y = (x + 4)^3$$

Worked Example

Sketch the graph of:

$$y = -(x + 4)^3$$

Sketch the graph of:

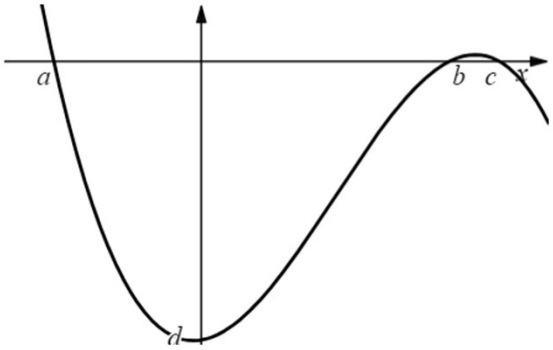
$$y = (4 - x)^3$$

T.62 4A: Qs 1-4, P.26 4.1 Qs 1-3

Worked Example

494b: **Identify the intercepts of a cubic with the coordinate axes, where the equation is factorised.**

The graph of $y = (x - 5)(6 - x)(x + 3)$ is sketched below.



a, b, c and d are the intercepts between the curve and the axes.

Find the values of a, b, c and d.

Worked Example

A curve is a positive cubic, touches the x-axis at 3 and crosses the x-axis at -2. Write a possible equation for the curve.

T.62 4A: Qs 5+, P.26 4.1 Qs 4+

4.2) Quartic graphs

Worked Example

Sketch the graph of:

$$y = (x + 3)(x + 4)(x - 3)(x - 4)$$

Sketch the graph of:

$$y = x(x - 3)^2(2 - x)$$

Worked Example

Sketch the graph of:

$$y = (x+2)^2(x-4)^2$$

Sketch the graph of:

$$y = x(x-4)(x+5)(x+6)$$

Worked Example

Sketch the graph of:

$$y = (x + 4)^2(x - 5)(6 - x)$$

Sketch the graph of:

$$y = (x - 2)(x + 2)^3$$

Worked Example

Sketch the graph of:

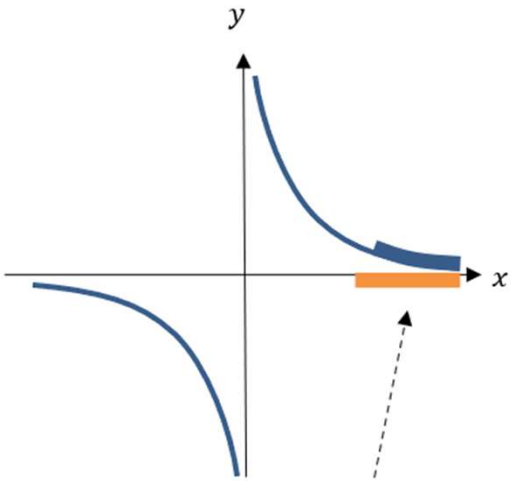
$$y = (x + 3)^4$$

Sketch the graph of:

$$y = -(x - 4)(x + 2)^3$$

4.3) Reciprocal graphs

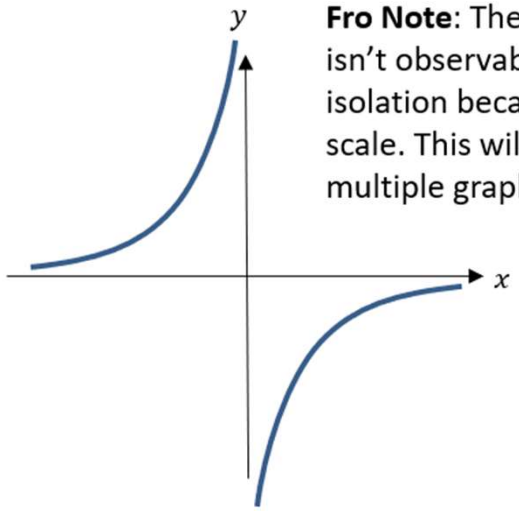
Sketch $y = \frac{1}{x}$



Notice the distance between this line and the x-axis (i.e. the line $y = 0$) gradually decreases as the lines go off towards infinity. The line $y = 0$ is known as an **asymptote** of the graph.

An asymptote is a line which the graph approaches but never reaches.

Sketch $y = -\frac{3}{x}$



Fro Note: The scaling caused by the 3 isn't observable for this graph in isolation because the axes have no scale. This will only be observation for multiple graphs on the same axes.

Asymptotes of $y = \frac{a}{x}$:

$y = 0,$

$x = 0$

T.67 4C: all Qs, P.28 4.3: all Qs

Notes

For a graph with equation $y = \frac{ax+b}{cx+d}$

Graph Feature		Technique
x-intercept	$-\frac{b}{a}$	Make y 0 (hence numerator 0).
y-intercept	$\frac{b}{d}$	Make x 0.
Vertical asymptote	$x = -\frac{d}{c}$	Make denominator 0.
Horizontal asymptote	$y = \frac{a}{c}$	Ignore constant terms (b and d).

You **shouldn't need to memorise these results**. Just use of intuition of what each feature means and therefore how we'd determine it.

Worked Example

493a: Identify the x and y intercepts of a more general reciprocal function.

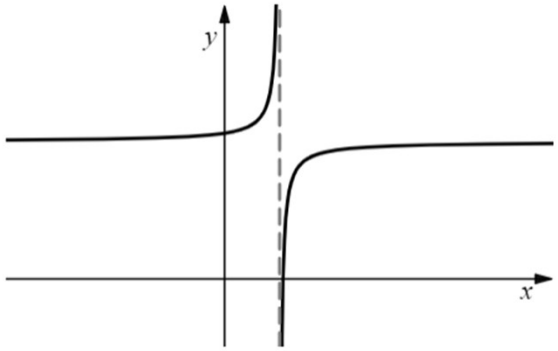
Determine the coordinates of the point where the curve with equation $y = 2 - \frac{5}{2x+3}$ crosses the y-axis.

(,)

Worked Example

493b: **Identify the equation of the vertical asymptote of a more general reciprocal function.**

The graph of $y = 4 - \dfrac{1}{5x - 4}$ is shown below:

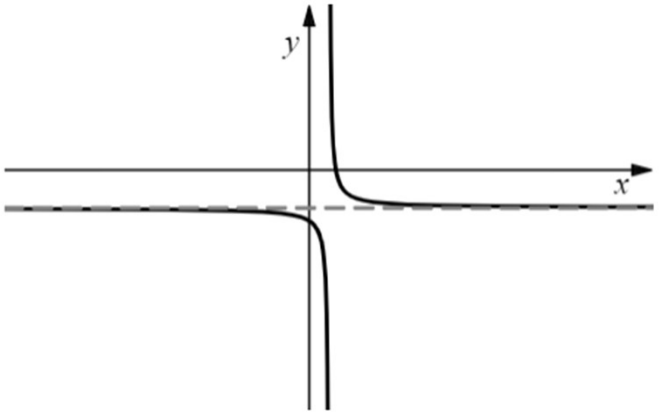


Find the equation of the vertical asymptote.

Worked Example

493c: **Identify the equation of the horizontal asymptote of a more general reciprocal function.**

The graph of $y = \dfrac{3x-1}{1-4x}$ is shown below:



Find the equation of the horizontal asymptote.

Worked Example

493g: Determine coefficients in a mo general reciprocal function that lead to a given x or y intercept.

A curve has equation

$$y = \frac{ax - b}{cx + d} \qquad x \neq -\frac{d}{c}$$

where a, b and d are positive prime numbers and c is a positive integer.

The curve crosses the x-axis at the point $\left(\frac{5}{7}, 0\right)$

Find the values of a and b.

Worked Example

493h: Determine coefficients in a more general reciprocal function that lead to a given vertical or horizontal asymptote.

A curve has equation

$$y = \frac{ax + b}{c - dx} \qquad x \neq \frac{c}{d}$$

where a, c and d are positive prime numbers and b is a positive integer.

An asymptote to the curve has equation $y = -\frac{5}{2}$

Find the values of a and d.

4.4) Points of intersection

Notes

Worked Example

On the same diagram sketch the curves with equations $y = -x^2(5x - a)$ and $y = -\frac{b}{x}$, where a, b are positive constants.

State, giving a reason, the number of real solutions to the equation $x^2(5x - a) + \frac{b}{x} = 0$

Worked Example

On the same diagram sketch the curves with equations $y = \frac{3}{x^2}$ and $y = x^2(x - 4)$.

State, giving a reason, the number of real solutions to the equation $x^4(x - 4) - 3 = 0$

T.69 4D: Qs1-2, P.29 4.4: Qs 2-3

Worked Example

On the same diagram sketch the curves with equations $y = x(x - 5)$ and $y = x(x - 3)^2$, and hence find the coordinates of any points of intersection.

Worked Example

Work out the range of values of a such that the graphs of $y = x^2 + a$ and $3y = x - 2$ have two points of intersection

T.69 4D: Qs3+, P.29 4.4: Qs 1, 4+

4.5) Translating graphs

This is all you need to remember when considering how transforming your function transforms your graph...

	Affects which axis?	What we expect or opposite?
Change **inside** $f(\,)$	x	Opposite
Change **outside** $f(\,)$	y	Direct

Therefore...

$y = f(x-3)$ ⟹ Translation by $\begin{pmatrix}3\\0\end{pmatrix}$

$y = f(x)+4$ ⟹ Translation by $\begin{pmatrix}0\\4\end{pmatrix}$

Effect of transformation on specific points

$y = f(x)$	$(4,3)$	$(1,0)$	$(6,-4)$
$y = f(x+1)$	$(3,3)$	$(0,0)$	$(5,-4)$
$y = f(x)-1$	$(4,2)$	$(1,-1)$	$(6,-5)$

T.74 4E: Qs 1,2, P.30 4.5: Qs 1,2

Notes

Worked Example

The point with coordinates $(-1.5, 0)$ lies on the curve with equation

$$y = (x + a)^3 + 6(x + a)^2 + 9(x + a)$$

where a is a constant. Find the two possible values of a

Worked Example

Sketch $y = x(x - 3)$. On the same axes, sketch $y = (x + a)(x + a - 3)$, where $a > 3$.

T.74 4E: Qs 3+, P.30 4.5: Qs 3+

4.6) Stretching graphs

This is all you need to remember when considering how transforming your function transforms your graph...

	Affects which axis?	What we expect or opposite?
Change **inside** $f(\,)$	x	Opposite
Change **outside** $f(\,)$	y	Direct

Therefore...

$y = f(5x)$ ⟹ Stretch in x-direction by scale factor $\frac{1}{5}$

$y = 2f(x)$ ⟹ Stretch in y-direction by scale factor 2

$\boldsymbol{y = f(x)}$	$\boldsymbol{(4,3)}$	$\boldsymbol{(1,0)}$	$\boldsymbol{(6,-4)}$
$y = f(2x)$	$(2,3)$	$(0.5,0)$	$(3,-4)$
$y = 3f(x)$	$(4,9)$	$(1,0)$	$(6,-12)$
$y = f(-x)$	$(-4,3)$	$(-1,0)$	$(-6,-4)$
$y = -f(x)$	$(4,-3)$	$(1,0)$	$(6,4)$

Notes

Notes

Worked Example

450e: Understand the effect on a point under the transformation $y = af(x)$

The curve with equation $y = f(x)$ has the maximum point $P(8, -14)$.

Find the image of P on the curve with equation $y = \frac{1}{2}f(x)$

450f: Understand the effect on a point under the transformation $y = f(ax)$

The point $P(6, -3)$ lies on the curve with equation $y = f(x)$.

Find the image of P on the curve with equation $y = f(3x)$

Worked Example KS 450i

The graph of $y = f(x)$ is shown in Figure 1.

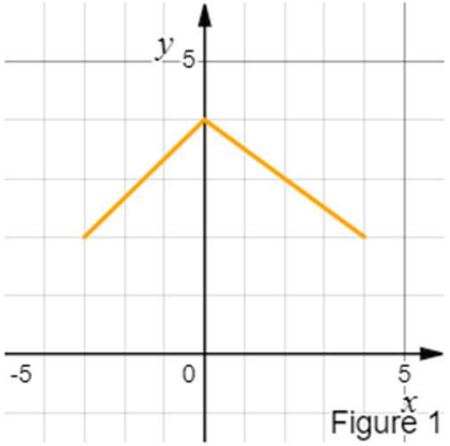


Figure 1

The graph of $y = af(x)$ is shown in Figure 2.

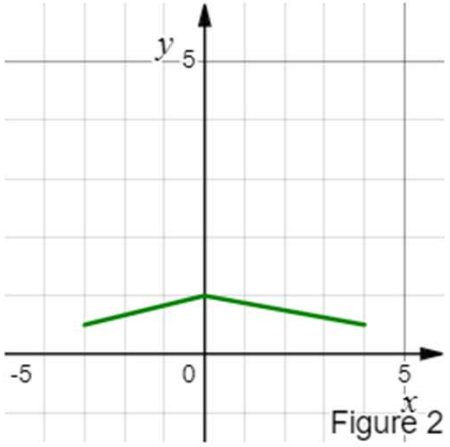


Figure 2

Determine the value of a.

T.78 4F: Qs 4+, P.32 4.6: Qs 4+

AS 2022 Q7

The curve C has equation

$$y = \frac{k^2}{x} + 1 \qquad x \in \mathbb{R},\ x \neq 0$$

where k is a constant.

(a) Sketch C stating the equation of the horizontal asymptote.

(3)

The line l has equation $y = -2x + 5$

(b) Show that the x coordinate of any point of intersection of l with C is given by a solution of the equation

$$2x^2 - 4x + k^2 = 0$$

(2)

(c) Hence find the exact values of k for which l is a tangent to C.

(3)

Your Turn

7. (*a*) Factorise completely $4x - x^3$

(2)

The curve C has equation

$$y = 4x - x^3$$

(*b*) Sketch C showing the coordinates of the points at which the curve cuts the x-axis.

(2)

The line l has equation $y = k$ where k is a constant.

Given that C and l intersect at 3 distinct points,

(*c*) find the range of values for k, writing your answer in set notation.

Solutions relying on calculator technology are not acceptable.

(3)

(Total for Question 7 is 7 marks)

4.7) Transforming functions

Important FACTS to LEARN:

- OUTSIDE THE BRACKET is DIRECT and applies to the $x - axis/direction$
- INSIDE THE BRACKET is OPPOSITE and applies to the y $- axis/direction$
- Summary/examples:

4 The graph of $y = f(x) + a$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.

5 The graph of $y = f(x + a)$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.

6 When you translate a function, any asymptotes are also translated.

7 The graph of $y = af(x)$ is a stretch of the graph $y = f(x)$ by a scale factor of a in the vertical direction.

8 The graph of $y = f(ax)$ is a stretch of the graph $y = f(x)$ by a scale factor of $\frac{1}{a}$ in the horizontal direction.

9 The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

10 The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

- *This is also true about ORDER of operations!* i.e. $f(2x - 1)$ is a translation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ followed by a stretch scale factor $\frac{1}{2}$ in the x direction.
- Where as $2f(x) - 1$ is a stretch scale factor 2 in the y direction followed by a translation $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Notes

Worked Example

Find the new coordinates under the transformations

$\boldsymbol{y = f(x)}$	$\boldsymbol{(-6, 4)}$	$\boldsymbol{(0, 1)}$
$y = f(x + 2)$		
$y = f(x) - 2$		
$y = f(3x)$		
$y = 4f(x)$		
$y = f\left(\frac{x}{5}\right)$		
$y = 6f(x)$		
$y = -f(x)$		
$y = f(-x)$		
$y = f(1 - x)$		

Worked Example

The point $A(2, 5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve:

a) $y = 2f(x) + 3$
b) $y = 3f(x) - 2$
c) $y = f(2x) + 3$
d) $y = f(3x) - 2$
e) $y = -f(x) + 3$
f) $y = -f(x) - 2$
g) $y = f(-x) + 3$
h) $y = f(-x) - 2$

Worked Example

The point $A(2, 5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve:

a) $y = -2f(x) + 3$
b) $y = -3f(x) - 2$
c) $y = 2f(-x) + 3$
d) $y = 3f(-x) - 2$
e) $y = -2f(-x) + 3$
f) $y = -3f(-x) - 2$
g) $y = 3f(2x) + 7$
h) $y = 7f(5x) - 2$
i) $y = -3f(2x) + 7$
j) $y = -7f(5x) - 2$
k) $y = 3f(x) + 7$
l) $y = -(3x + 1) - 2$

T.80 4G: all Qs, P.33 4.7 all Qs

AS 2019

11. $f(x) = 2x^3 - 13x^2 + 8x + 48$

(a) Prove that $(x - 4)$ is a factor of $f(x)$.

(2)

(b) Hence, using algebra, show that the equation $f(x) = 0$ has only two distinct roots.

(4)

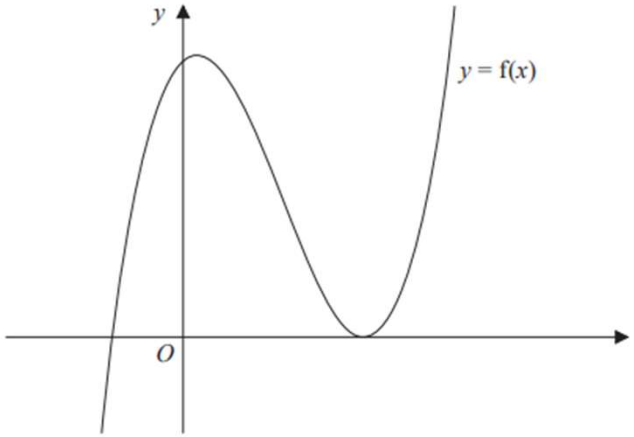


Figure 2

Figure 2 shows a sketch of part of the curve with equation $y = f(x)$.

(c) Deduce, giving reasons for your answer, the number of real roots of the equation

$$2x^3 - 13x^2 + 8x + 46 = 0$$

(2)

Given that k is a constant and the curve with equation $y = f(x + k)$ passes through the origin,

(d) find the two possible values of k.

(2)

Your Turn

11.

$$f(x) = x^3 + x^2 - 8x - 12$$

(a) Prove that $(x - 3)$ is a factor of $f(x)$.

(2)

(b) Hence, using algebra, show that the equation $f(x) = 0$ has only 2 distinct roots.

(3)

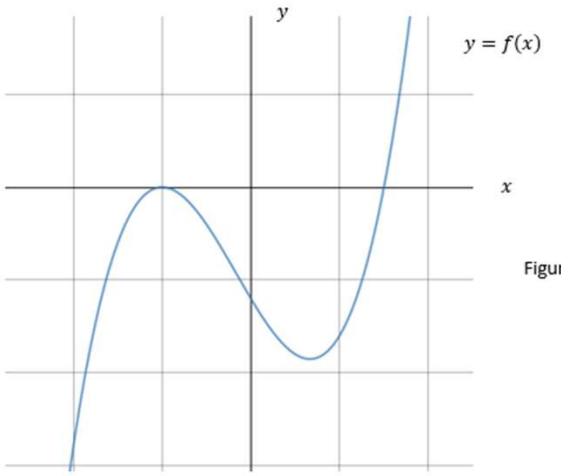


Figure B

Figure B shows a sketch of part of the curve $y = f(x)$.

(c) Deduce, giving reasons for your answer, the number of real roots of the equation

$$2x^3 + 2x^2 - 16x - 50 = 0$$

(3)

Given that k is a constant and the curve with equation $y = f(kx)$ passes through (1,0),

(d) Find the two possible values of k

(2)

AS Specimen

Graphs & Transformations

4.

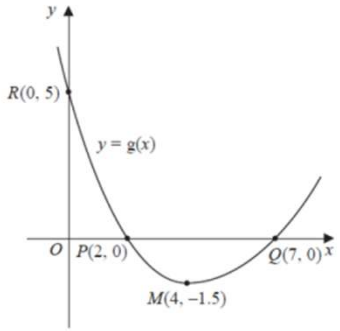


Figure 1

Figure 1 shows a sketch of the curve with equation $y = g(x)$.

The curve has a single turning point, a minimum, at the point $M(4, -1.5)$.

The curve crosses the x-axis at two points, $P(2, 0)$ and $Q(7, 0)$.

The curve crosses the y-axis at a single point $R(0, 5)$.

(a) State the coordinates of the turning point on the curve with equation $y = 2g(x)$.

(1)

(b) State the largest root of the equation $g(x + 1) = 0$.

(1)

(c) State the range of values of x for which $g'(x) \leq 0$.

(1)

Given that the equation $g(x) + k = 0$, where k is a constant, has no real roots,

(d) state the range of possible values for k.

(1)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Question	Scheme	Marks	AOs
4(a)	$(4, -3)$	B1	1.2
		(1)	
(b)	$k > 1.5$	B1	2.5,3g
		(1)	
(c)	$x \leqslant 4$	B1 [not incl]	1.1b
		(1)	
(d)	$k = 0$	B1	1.1b
		(1)	

(4 marks)

Summary of Key Points

Summary of key points

1. If p is a root of the function $f(x)$, then the graph of $y = f(x)$ touches or crosses the x-axis at the point $(p, 0)$.
2. The graphs of $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$, where k is a real constant, have asymptotes at $x = 0$ and $y = 0$.
3. The x-coordinate(s) at the points of intersection of the curves with equations $y = f(x)$ and $y = g(x)$ are the solution(s) to the equation $f(x) = g(x)$.
4. The graph of $y = f(x) + a$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.
5. The graph of $y = f(x + a)$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.
6. When you translate a function, any asymptotes are also translated.
7. The graph of $y = af(x)$ is a stretch of the graph $y = f(x)$ by a scale factor of a in the vertical direction.
8. The graph of $y = f(ax)$ is a stretch of the graph $y = f(x)$ by a scale factor of $\frac{1}{a}$ in the horizontal direction.
9. The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.
10. The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

T.82 mixed ex P.35 BSG