KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Pure Mathematics
13 Integration

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

Prior knowledge check

1 Simplify these expressions

a $\dfrac{x^3}{\sqrt{x}}$ **b** $\dfrac{\sqrt{x} \times 2x^3}{x^2}$

c $\dfrac{x^3 - x}{\sqrt{x}}$ **d** $\dfrac{\sqrt{x} + 4x^3}{x^2}$

← **Sections 1.1, 1.4**

2 Find $\dfrac{dy}{dx}$ when y equals

a $2x^3 + 3x - 5$ **b** $\frac{1}{2}x^2 - x$

c $x^2(x + 1)$ **d** $\dfrac{x - x^5}{x^2}$

← **Section 12.5**

3 Sketch the curves with the following equations:

a $y = (x + 1)(x - 3)$

b $y = (x + 1)^2(x + 5)$

← **Chapter 4**

13.1) Integrating x^n

Integration is the inverse of differentiation.

i.e.
differentiation: $ax^n \xrightarrow{d/dx} n \cdot ax^{n-1}$

i.e. (1) × original power ($\times n$)
(2) subtract one from power ($n-1$)

So inverting
(1) add one to power ($n+1$)
(2) ÷ new power ($\div n+1$)
[or $\times \frac{1}{n+1}$]

now: $ax^n \rightarrow \frac{1}{n+1} \cdot ax^{n+1} + c$

d/dx backwards to 0!

Eg If $\frac{dy}{dx} = 5x^2 + 7x^{-2} - \frac{1}{2}x^9 + 11x^0$

then $y = \frac{1}{3} \times 5x^3 + \frac{1}{-1} \times 7x^{-1} - \frac{1}{10} \times \frac{1}{2}x^{10} + \frac{1}{1} \times 11x^1 + c$

$\Rightarrow y = \frac{5}{3}x^3 - 7x^{-1} - \frac{1}{20}x^{10} + 11x + c$

Notes

13.2) Indefinite integrals

The following notation could be used to differentiate an expression:

The dx here means differentiating "with respect to x".

$$\frac{d}{dx}(5x^2) = 10x$$

There is similarly notation for integrating an expression:

$$\int 10x \; dx = 5x^2 + c$$

"Integrate…" "…this expression" "…with respect to x"

(the dx is needed just as it was needed in the differentiation notation at the top of this slide)

This is known as **indefinite integration**, in contrast to definite integration, which we'll see later in the chapter.

It is called 'indefinite' because the exact expression is unknown (due to the $+c$).

Notes

When you are integrating a polynomial function, you can integrate the terms one at a time.

- $\int(\mathbf{f}(x) + \mathbf{g}(x))\,\mathrm{d}x = \int \mathbf{f}(x)\,\mathrm{d}x + \int \mathbf{g}(x)\,\mathrm{d}x$

Example 4

Find:

a $\int(x^{\frac{1}{2}} + 2x^3)\,\mathrm{d}x$ **b** $\int(x^{-\frac{3}{2}} + 2)\,\mathrm{d}x$ **c** $\int(p^2x^{-2} + q)\,\mathrm{d}x$ **d** $\int(4t^2 + 6)\,\mathrm{d}t$

a $\int(x^{\frac{1}{2}} + 2x^3)\mathrm{d}x = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2x^4}{4} + c$ — First apply the rule term by term.

$= \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^4 + c$ — Simplify each term.

b $\int(x^{-\frac{3}{2}} + 2)\mathrm{d}x = \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + 2x + c$ — Remember $-\frac{3}{2} + 1 = -\frac{1}{2}$ and the integral of the constant 2 is $2x$.

$= -2x^{-\frac{1}{2}} + 2x + c$

c $\int(p^2x^{-2} + q)\mathrm{d}x = \frac{p^2}{-1}x^{-1} + qx + c$ — The dx tells you to integrate with respect to the variable x, so any other letters must be treated as constants.

$= -p^2x^{-1} + qx + c$

d $\int(4t^2 + 6)\mathrm{d}t = \frac{4t^3}{3} + 6t + c$ — The dt tells you that this time you must integrate with respect to t. Use the rule for integrating x^n but replace x with t: If $\frac{\mathrm{d}y}{\mathrm{d}t} = kt^n$, then $y = \frac{k}{n+1}t^{n+1} + c, n \neq -1$.

Example 5

Find:

a $\int\left(\frac{2}{x^3} - 3\sqrt{x}\right)\mathrm{d}x$ **b** $\int x\left(x^2 + \frac{2}{x}\right)\mathrm{d}x$ **c** $\int\left((2x)^2 + \frac{\sqrt{x}+5}{x^2}\right)\mathrm{d}x$

a $\int\left(\frac{2}{x^3} - 3\sqrt{x}\right)\mathrm{d}x$

$= \int(2x^{-3} - 3x^{\frac{1}{2}})\mathrm{d}x$ — First write each term in the form x^n.

$= \frac{2}{-2}x^{-2} - \frac{3}{\frac{3}{2}}x^{\frac{3}{2}} + c$ — Apply the rule term by term.

$= -x^{-2} - 2x^{\frac{3}{2}} + c$ — Simplify each term.

$= -\frac{1}{x^2} - 2\sqrt{x^3} + c$ — Sometimes it is helpful to write the answer in the same form as the question.

b $\int x\left(x^2 + \frac{2}{x}\right)\mathrm{d}x$

$= \int(x^3 + 2)\mathrm{d}x$ — First multiply out the bracket.

$= \frac{x^4}{4} + 2x + c$ — Then apply the rule to each term.

c $\int\left((2x)^2 + \frac{\sqrt{x}+5}{x^2}\right)\mathrm{d}x$

$= \int\left(4x^2 + \frac{x^{\frac{1}{2}}}{x^2} + \frac{5}{x^2}\right)\mathrm{d}x$ — Simplify $(2x)^2$ and write $\sqrt{x}$ as $x^{\frac{1}{2}}$.

$= \int(4x^2 + x^{-\frac{3}{2}} + 5x^{-2})\mathrm{d}x$ — Write each term in the form x^n.

$= \frac{4}{3}x^3 + \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{5x^{-1}}{-1} + c$ — Apply the rule term by term.

$= \frac{4}{3}x^3 - 2x^{-\frac{1}{2}} - 5x^{-1} + c$

$= \frac{4}{3}x^3 - \frac{2}{\sqrt{x}} - \frac{5}{x} + c$ — Finally simplify the answer.

Question 1

Find

$$\int \left(4x^7 - x^6 \right) \, dx$$

Question 2

Find

$$\int \left(-5x^6 + 4x^5 + 2x^2 + x^{-2} \right) \, dx$$

Question 3

Find

$$\int \left(5x^8 + 4x^4 - 4x + 4 \right) \, dx$$

Question 4

Find

$$\int \left(5x^6 + 3x^4 + 2 + 4x^{-4} \right) \, dx$$

Question 5

Find

$$\int \left(-5x^5 + 4x^4 + x^3 + \frac{5}{4}x^{-\frac{7}{4}} \right) \, dx$$

Question 6

Find

$$\int \left(4x^5 + \frac{2}{\sqrt{x}} + \frac{2}{x^{\frac{3}{4}}} \right) \, d\,x$$

Question 7

Find

$$\int \left(\frac{4}{3}\sqrt[3]{x} + \frac{3}{2}\sqrt{x} + \frac{4}{\sqrt{x}} - \frac{4}{x^7} \right) \, d\,x$$

Question 8

Find

$$\int \left(-x + 4\sqrt{x} + \frac{1}{x^3} \right) \, d\,x$$

Question 9

Find

$$\int \left(\sqrt{x} + \frac{5}{\sqrt{x}} \right) \, d\,x$$

Question 10

Find

$$\int \left(5x^7 + 4x^2 + \frac{2}{\sqrt[3]{x}} + 3\sqrt[3]{x} \right) \, d\,x$$

Question 11

Find

$$\int 5x^4\left(-4x^2+5x^3\right)\ d\,x$$

Question 12

Find

$$\int \left(2x^3-2x\right)\left(2x^4+5x^5\right)\ d\,x$$

$I =$ .... $+c$

Question 13

Find

$$\int \left(4x^3+3x\right)\left(2x^3+2x^5\right)\ d\,x$$

Question 14

Find

$$\int \left(5x^3+4x^{-2}\right)\left(-5x^3+4x^{-3}\right)\ d\,x$$

Question 15

Find

$$\int 5x^4\left(3x^{-2}+4x^3\right)\ d\,x$$

Question 16

Find

$$\int \left(\frac{2x^{-1}+4+x^{-2}}{5x^2}\right)\ d\,x$$

Question 17

Find

$$\int \left(\frac{4x^{-\frac{1}{3}}+4x^3-3x^{\frac{1}{2}}}{3x^{-1}}\right)\ d\,x$$

Question 18

Find

$$\int \left(\frac{2x^2+x^{-\frac{1}{2}}}{3x}\right)\ d\,x$$

Question 19

Find $\int \left(\frac{3x^7+x}{x^4}\right)\ d\,x$

Question 20

Find $\int \left(\frac{-5x^{-\frac{1}{2}}+4x^6-4x^{\frac{1}{3}}}{4x^3}\right)\ d\,x$

Worked Example

$$\int \left(\frac{p}{2x^2} + pq\right)\, dx = \frac{2}{x} + 12 + c$$

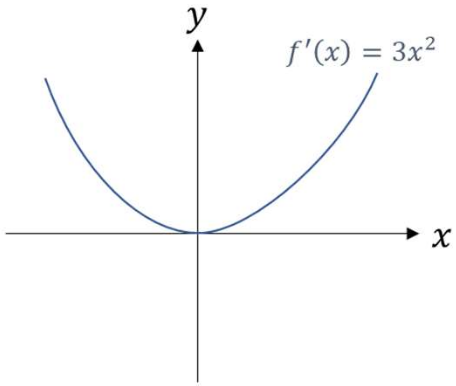
Find the value of p and the value of q

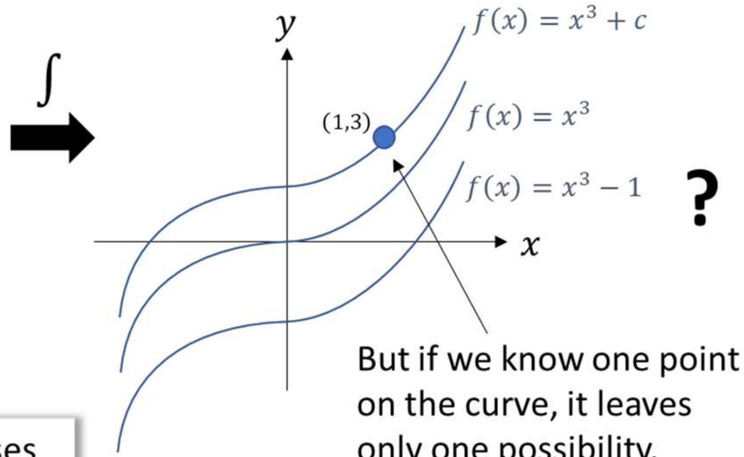
T.291: Ex 13B Qs10-15, P.105: Qs5-7

13.3) Finding functions

Need one co-ordinate to determine "+c"

Recall that when we integrate, we get a constant of integration, which could be any real value. This means **we don't know what the exact original function was**.





> The curve with equation $y = f(x)$ passes through (1,3). Given that $f'(x) = 3x^2$, find the equation of the curve.

$$f'(x) = 3x^2$$
$$\therefore f(x) = x^3 + c$$
Using the point (1,3): $3 = 1^3 + c \quad \therefore c = 2$
$$f(x) = x^3 + 2$$

Notes

1 **Example 6**

The curve C with equation $y = \mathrm{f}(x)$ passes through the point (4, 5). Given that $\mathrm{f}'(x) = \frac{x^2 - 2}{\sqrt{x}}$, find the equation of C.

$\mathrm{f}'(x) = \frac{x^2 - 2}{\sqrt{x}} = x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}$ — First write f′(x) in a form suitable for integration.

So $\mathrm{f}(x) = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$ — Integrate as normal and don't forget the + c.

$= \frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + c$

But $\mathrm{f}(4) = 5$ — Use the fact that the curve passes through (4, 5).

So $5 = \frac{2}{5} \times 2^5 - 4 \times 2 + c$ — Remember $4^{\frac{5}{2}} = 2^5$.

$5 = \frac{64}{5} - 8 + c$

$5 = \frac{24}{5} + c$

So $c = \frac{1}{5}$ — Solve for c.

So $y = \frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + \frac{1}{5}$ — Finally write down the equation of the curve.

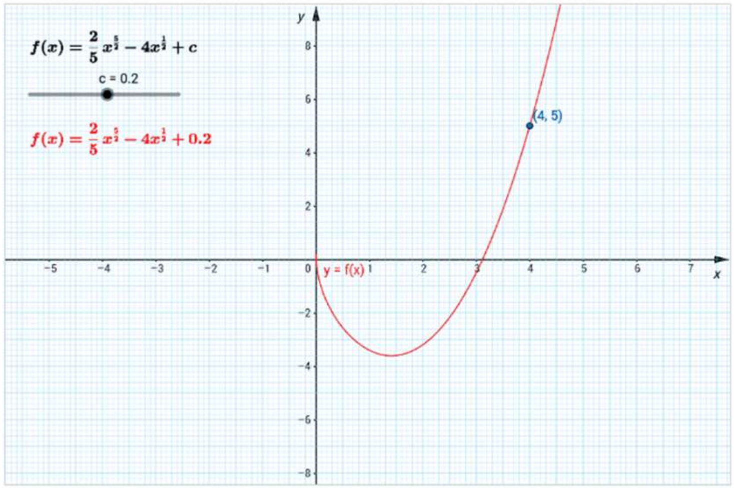
Online Explore the solution using GeoGebra.

13 Integration, page 294, example 6 – GeoGebra

13 Integration, page 294, example 6

Author: Pearson Secondary Maths

Explore how the value of c affects the curve.



Worked Example

522a: **Determine a function by integrating $\frac{dy}{dx}$**

Find the equation of the curve given that $\frac{dy}{dx} = 5x^3 + 9x^2 + 5x$ and that the curve passes through the point $(3, 231)$

T.291: Ex 13C Qs all, P.105: Qs all

7. Given that k is a positive constant and $\int_1^k \left(\frac{5}{2\sqrt{x}} + 3\right) dx = 4$

(a) show that $3k + 5\sqrt{k} - 12 = 0$

(4)

(b) Hence, using algebra, find any values of k such that

$$\int_1^k \left(\frac{5}{2\sqrt{x}} + 3\right) dx = 4$$

(4)

Your Turn

7 Given that k is a positive constant and $\int_1^k \left(\frac{7}{2\sqrt{x}} + 2\right) dx = 21$

(*a*) show that $2k + 7\sqrt{k} - 30 = 0$

(4)

(*b*) Hence, using algebra, find any values of k such that

$$\int_1^k \left(\frac{7}{2\sqrt{x}} + 2\right) dx = 21$$

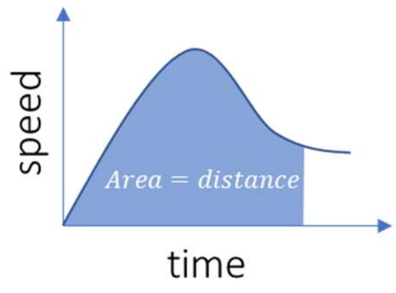
(4)

(Total for Question 7 is 8 marks)

13.4) Definite integrals

So far we've seen integration as '**the opposite of differentiation**', allowing us to find $y = f(x)$ when we know the gradient function $y = f'(x)$.

In practical settings however the most useful use of integration is that **it finds the area under a graph**. Remember at GCSE for example when you estimated the area under a speed-time graph, using trapeziums, to get the distance?
If you knew the equation of the curve, you could get the exact area!

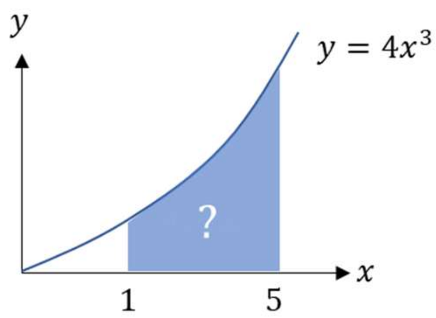


Before we do this, we need to understand how to find a **definite integral**:

These are known as **limits**, which give the values of x we're finding the area between.

We integrate as normal, but put expression in **square brackets**, meaning **we still need to evaluate the integrated expression using the limits**.

$$\int_1^5 4x^3\,dx = [x^4]_1^5$$
$$= (5^4) - (1^4)$$
$$= 624$$



Write (...) − (...) and evaluate the expression for each of the limits, top one first.

Notes

Worked Example

525a **Evaluate a definite integral where the integrand is a collection of terms in the form ax^b**

Evaluate

$$\int_1^5 (-x^3 + 6x^2 + 4x)\,dx$$

$$= \left[-\frac{1}{4}x^4 + 2x^3 + 2x^2\right]_1^5$$

← put 'integrated function inside square [...] brackets. limits

$$= \left(-\frac{1}{4}(5)^4 + 2(5)^3 + 2(5)^2\right)$$

← (put top limit in 1st)

$$- \left(-\frac{1}{4}(1)^4 + 2(1)^3 + 2(1)^2\right)$$

− (bottom limit)

$$= \frac{575}{4} - \frac{15}{4} = 140$$

↑ Set out like this for harder Qs when you do exact form next year.

T.297: Ex 13D Qs 1-3, P.106: Qs 1-5

525

Question 1

Evaluate $\int_4^6 \left(x^3+9x^2+4x\right)\,dx$

Question 2

Evaluate $\int_2^5 \left(4x^3+3x^2+4\right)\,dx$

Question 3

Evaluate $\int_4^9 \left(3x^3+9x^2+5x\right)\,dx$

Question 4

Evaluate $\int_1^4 \left(3x^2-3x\right)\,dx$

Question 5

Evaluate $\int_1^{16} \left(4x^3+2x+5\right)\,dx$

Question 6

Evaluate $\int_4^6 \left(9x^2+4x\right)\,dx$

Question 7

Evaluate $\int_1^9 \left(2x^3+9x^2+3x\right)\,dx$

Question 8

Evaluate $\int_1^{16} \left(4x^3+3x^2+5x\right)\,dx$

Question 9

Use algebraic integration to find the exact value of
$\int_4^7 \frac{3x^2+6\sqrt{x}}{4x}\,dx$

Question 10

Use algebraic integration to find the exact value of $\int_8^9 \frac{2\sqrt{x}+4}{x\ 2}\,dx$

Question 11

Use algebraic integration to find the exact value of $\int_3^6 \frac{x^2-2}{5x\ 2}\,dx$

Question 12

Use algebraic integration to find the exact value of
$\int_7^9 \frac{2x^2-6\sqrt{x}-1}{x\ 2}\,dx$

Question 13

Use algebraic integration to find the exact value of $\int_7^9 \frac{4x-3}{\sqrt{x}}\,dx$

Question 14

Use algebraic integration to find the exact value of

$\int_4^6 \frac{2x^2+3\sqrt{x}}{x}\,dx$

Question 15

Use algebraic integration to find the exact value of $\int_1^7 \frac{4\sqrt{x}+5x}{3x}\,dx$

Question 16

Use algebraic integration to find the exact value of $\int_1^4 \frac{4\sqrt{x}+6x}{2x}\,dx$

Worked Example

Given that P is a constant and

$$\int_3^7 (4Px + 7)\,dx = 108P^2$$

find the possible values of P

Worked Example

Given that $\int_1^k \frac{1}{\sqrt[4]{x}} dx = \frac{28}{3}$,

calculate the value of k

T.297: Ex 13D Qs 4+, P.106: Qs 6+

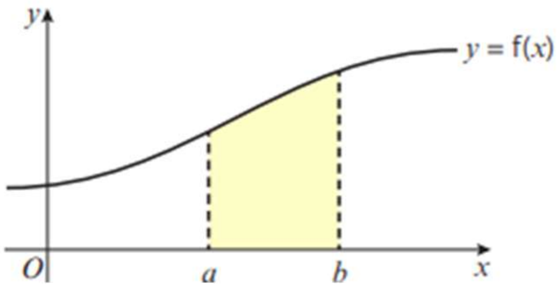
13.5) Areas under curves

WHAT YOU MUST KNOW:

The area between a positive curve, the x-axis and the lines $x = a$ and $x = b$ is given by

Area $= \int_a^b y \, dx$

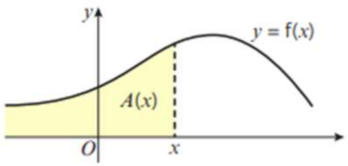
where $y = f(x)$ is the equation of the curve.



Background:

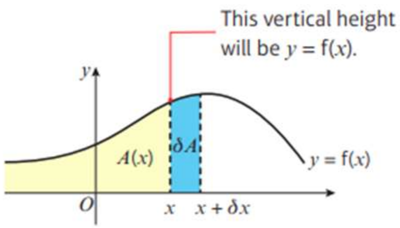
Definite integration can be used to find the area under a curve.

For any curve with equation $y = f(x)$, you can define the area under the curve to the left of x as a function of x called $A(x)$. As x increases, this area $A(x)$ also increases (since x moves further to the right).



If you look at a small increase in x, say δx, then the area increases by an amount $\delta A = A(x + \delta x) - A(x)$.

This increase in the δA is approximately rectangular and of magnitude $y\delta x$. (As you make δx smaller any error between the actual area and this will be negligible.)



So you have $\delta A \approx y\delta x$

or $\dfrac{\delta A}{\delta x} \approx y$

and if you take the limit $\lim_{\delta x \to 0}\left(\dfrac{\delta A}{\delta x}\right)$ then you will see that $\dfrac{dA}{dx} = y$.

Now if you know that $\dfrac{dA}{dx} = y$, then to find A you have to integrate, giving $A = \int y \, dx$.

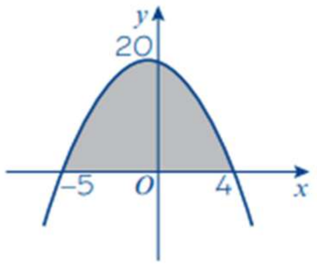
Notes

Worked Example

Find the area of the finite region between the curve with equation $y = 20 - x - x^2$ and the x-axis.

$y = 20 - x - x^2 = (4 - x)(5 + x)$

Factorise the expression.



Draw a sketch of the graph. $x = 4$ and $x = -5$ are the points of intersection of the curve and the x-axis.

$$\text{Area} = \int_{-5}^{4}(20 - x - x^2)\,dx$$

$$= \left[20x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-5}^{4}$$

$$= \left(80 - 8 - \frac{64}{3}\right) - \left(-100 - \frac{25}{2} + \frac{125}{3}\right)$$

$$= \frac{243}{2}$$

You don't normally need to give units when you are finding areas on graphs.

T.299: Ex 13E Qs 1,2, P.107: Qs 1-3

Worked Example

Find the area of the finite region bounded by the curve with equation $y = x^2(x+2)$ and the x-axis

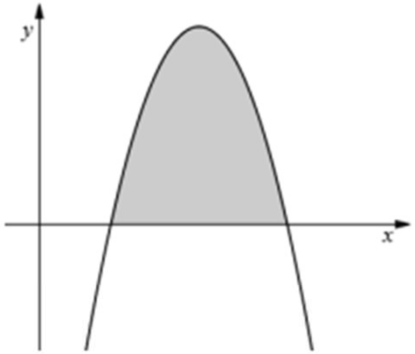
T.299: Ex 13E Qs 3+, P.107: Qs 4+

Question 1

The diagram below shows the graph of

$y = (7 - x)(x - 2)$

Find the exact area of the shaded region.

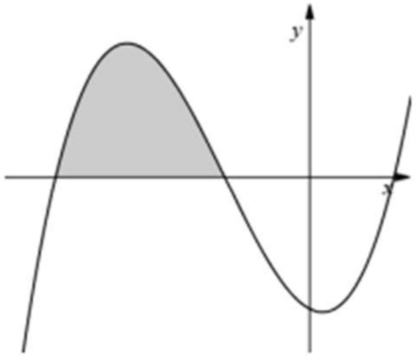


Question 2

The diagram below shows the graph of

$y = (x + 3)(x + 1)(x - 1)$

Find the exact area of the shaded region.

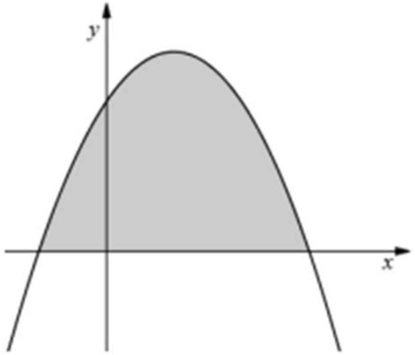


Question 3

The diagram below shows the graph of

$y = -x^2 + 2x + 3$

Find the exact area of the shaded region.

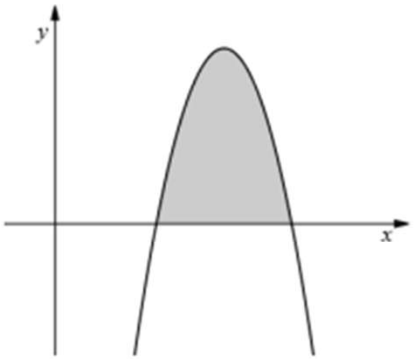


Question 4

The diagram below shows the graph of

$y = -x^2 + 10x - 21$

Find the exact area of the shaded region.

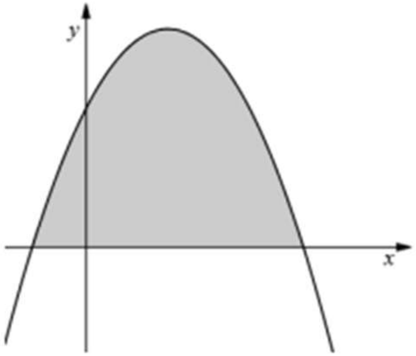


Question 5

The diagram below shows the graph of

$y = -x^2 + 3x + 4$

Find the exact area of the shaded region.

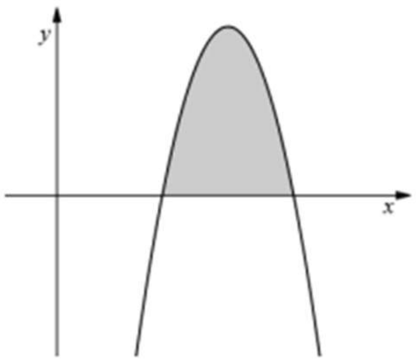


Question 6

The diagram below shows the graph of

$y = (9 - x)(x - 4)$

Find the exact area of the shaded region.



Question 7

Find, in terms of a, the value of $\int_4^a \left(5x^3 - 2x^{\frac{1}{2}}\right)\,dx$ where a is a constant greater than 4.

Question 8

Find, in terms of k, the value of $\int_3^k \left(4x^{-4} + x^{-\frac{1}{2}}\right)\,dx$ where k is a constant greater than 3.

Question 9

Find, in terms of k, the value of $\int_2^k 4x^{-\frac{1}{2}}\,dx$ where k is a constant greater than 2.

Question 10

Find, in terms of a, the value of $\int_2^a \left(4x^{-2} - 2x^3\right)\,dx$ where a is a constant greater than 2.

Question 11

Find, in terms of a, the value of $\int_2^a \left(3x^{-\frac{3}{2}} - 3x^{-2}\right)\,dx$ where a is a constant greater than 2.

Question 12

Find, in terms of a, the value of $\int_4^a 4x^3\,dx$ where a is a constant greater than 4.

13.6) Areas under the x-axis

- **When the area bounded by a curve and the x-axis is below the x-axis, $\int y\,dx$ gives a negative answer.**

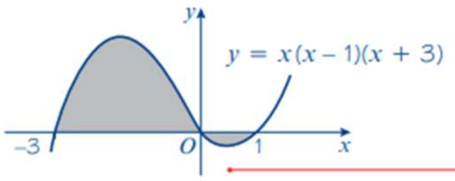
Sketch the curve with equation $y = x(x-1)(x+3)$ and find the area of the finite region bounded by the curve and the x-axis.

When $x = 0$, $y = 0$
When $y = 0$, $x = 0$, 1 or -3

Find out where the curve cuts the axes.

$x \to \infty$, $y \to \infty$
$x \to -\infty$, $y \to -\infty$

Find out what happens to y when x is large and positive or large and negative.



Problem-solving

Always draw a sketch, and use the points of intersection with the x-axis as the limits for your integrals.

The area is given by $\int_{-3}^{0} y\,dx - \int_{0}^{1} y\,dx$

Since the area between $x = 0$ and 1 is below the axis the integral between these points will give a negative answer.

Now $\int y\,dx = \int (x^3 + 2x^2 - 3x)dx$

Multiply out the brackets.

$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]$$

So $\int_{-3}^{0} y\,dx = (0) - \left(\frac{81}{4} - \frac{2}{3} \times 27 - \frac{3}{2} \times 9\right)$

$$= \frac{45}{4}$$

and $\int_{0}^{1} y\,dx = \left(\frac{1}{4} + \frac{2}{3} - \frac{3}{2}\right) - (0)$

$$= -\frac{7}{12}$$

So the area required is $\frac{45}{4} + \frac{7}{12} = \frac{71}{6}$

Watch out If you try to calculate the area as a single definite integral, the positive and negative areas will partly cancel each other out.

Notes

Worked Example

Find the total area bound between the curve $y = x(x-2)(x-4)$ and the x-axis.

T.301: Ex 13F Q 1, P.108: Qs 1,2

Worked Example

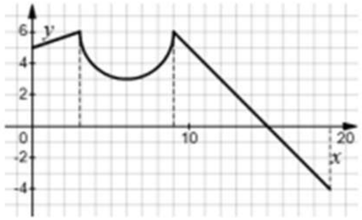
Find the total area bound between the curve $y = x^3 + 2x^2 - 15x$ and the x-axis.

T.301: Ex 13F Qs 2,3 , P.109: Qs 3+

525 pt 3

Question 1

The diagram shows the graph of $y = f(x)$, which is made up of line segments and a semicircle.

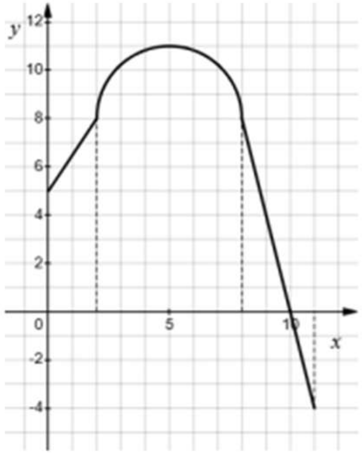


Find the exact value of

$\int_0^{19} f(x)\, dx$

Question 2

The diagram shows the graph of $y = f(x)$, which is made up of line segments and a semicircle.

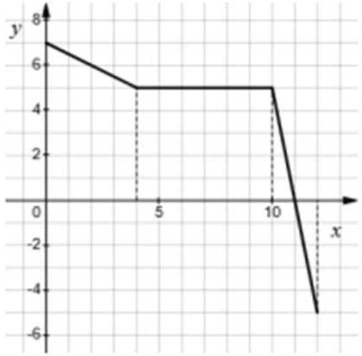


Find the exact value of

$\int_0^{11} f(x)\, dx$

Question 3

The diagram shows the graph of $y = f(x)$, which is made up of line segments.

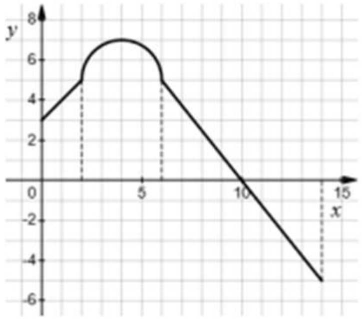


Find the exact value of

$\int_0^{12} f(x)\, dx$

Question 4

The diagram shows the graph of $y = f(x)$, which is made up of line segments and a semicircle.



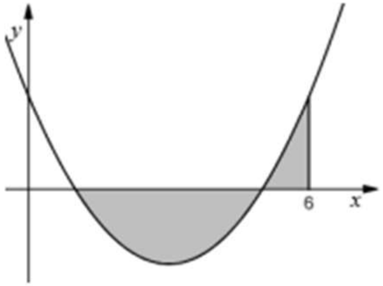
Find the exact value of

$\int_0^{14} f(x)\, dx$

Question 5

The diagram below shows the graph of

$y = x^2 - 6x + 5$

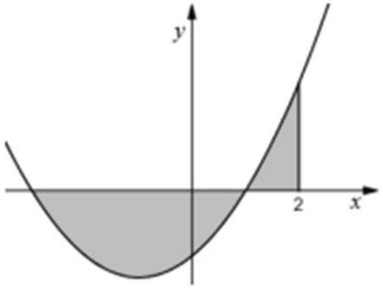


Find the exact value of the shaded area.

Question 6

The diagram below shows the graph of

$y = 3x^2 + 6x - 9$

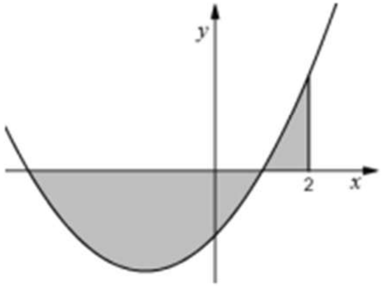


Find the exact value of the shaded area.

Question 7

The diagram below shows the graph of

$y = 2x^2 + 6x - 8$

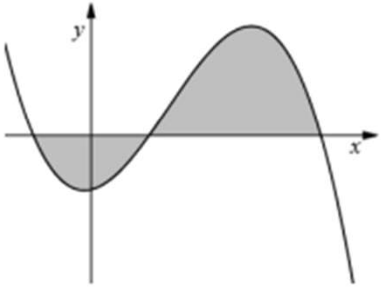


Find the exact value of the shaded area.

Question 8

The diagram below shows the graph of

$y = (x + 1)(1 - x)(x - 4)$

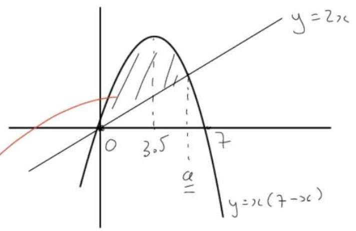


Find the exact value of the shaded area.

13.7) Areas between curves and lines

Determine the area bounded by the curve with equation $y = x(7 - x)$ and the line with equation $y = 2x$

① sketch:



do $\int \left(\begin{matrix}\text{line above}\\\text{shaded}\end{matrix}\right) dx - \int \left(\begin{matrix}\text{line below}\\\text{shaded}\end{matrix}\right) dx$

② $\int_0^a x(7-x)\,dx - \int_0^a 2x\,dx$

or $\equiv \int_0^a 7x - x^2 - 2x\,dx = \int_0^a 5x - x^2\,dx$

③ a : $2x = 7x - x^2$

$\Rightarrow x^2 - 5x = 0$

$x(x-5) = 0$

$\Rightarrow x = 0 \quad x = 5$

$\therefore a = 5$

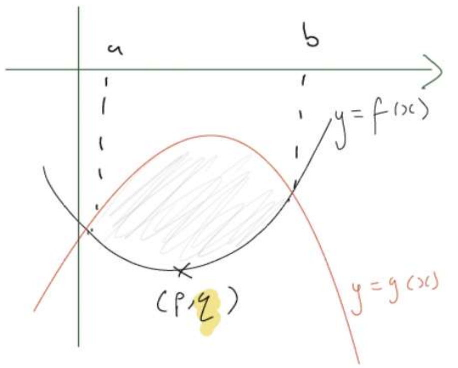
$= \int_0^5 5x - x^2\,dx$

$= \left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_0^5$

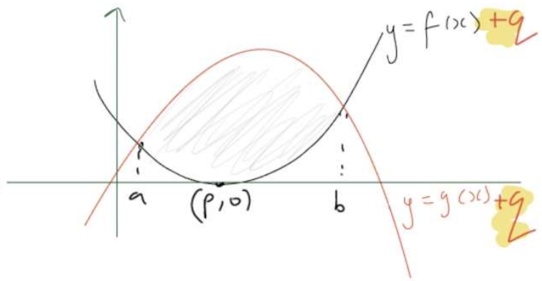
④ $= \left(\frac{5}{2}(5)^2 - \frac{1}{3}(5)^3\right) - (0 - 0) = \frac{125}{6}$

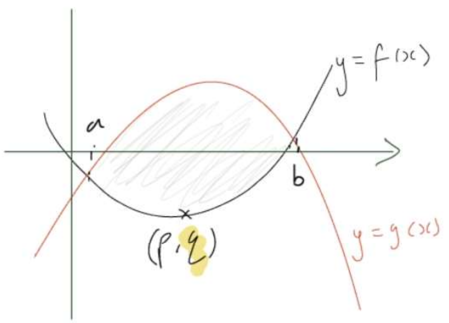
Areas under the axis between lines

Same method for these too, because:



If any shaded below – translate up using minimum point (p, q)





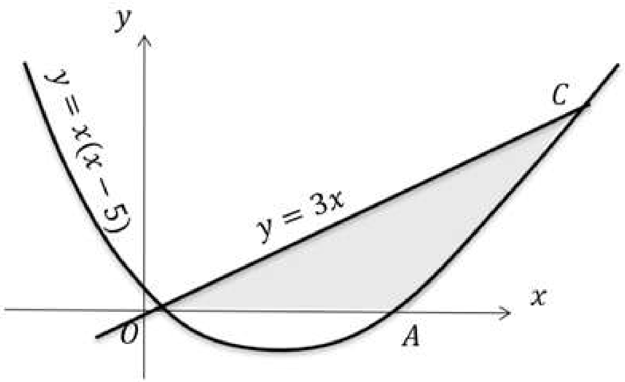
Now do as normal

$$\int_a^b (g(x) + q) - (f(x) + q) \, dx$$

$\Rightarrow$ same as $\displaystyle\int_a^b g(x) - f(x) \, dx$

Worked Example

The diagram shows a sketch of the curve with equation $y = x(x-5)$ and the line with equation $y = 3x$.
Find the area of the shaded region OAC.



T.304: Ex 13G Qs 1-3 , P.110: Qs 1,2

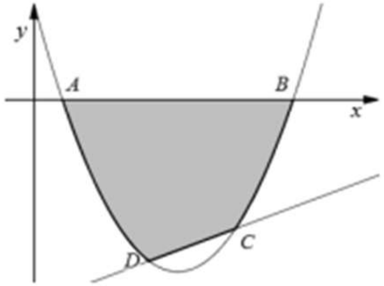
Worked Example

Determine the area bounded by the curve with equation $y = 5x - x^2 - 3$ and the line with equation $y = 5 - x$

T.304: Ex 13G Qs 4+ , P.110: Qs 3+

Question 9

The shaded shape $ABCD$ below is bounded by the x-axis, the curve $y = x^2 - 10x + 9$ and the line passing through points C and D.



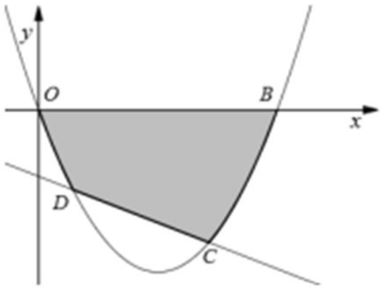
The coordinates of point C are $(7, -12)$.
The coordinates of point D are $(4, -15)$.

Find the exact value of the shaded area.

....

Question 10

The shaded shape $ABCD$ below is bounded by the x-axis, the curve $y = x^2 - 7x$ and the line passing through points C and D.



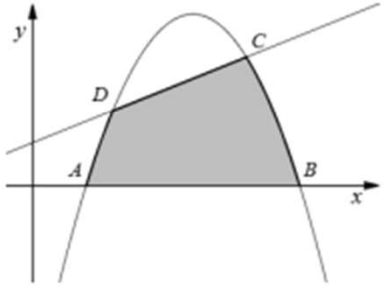
The coordinates of point C are $(5, -10)$.
The coordinates of point D are $(1, -6)$.

Find the exact value of the shaded area.

....

Question 11

The shaded shape $ABCD$ below is bounded by the x-axis, the curve $y = -x^2 + 12x - 20$ and the line passing through points C and D.



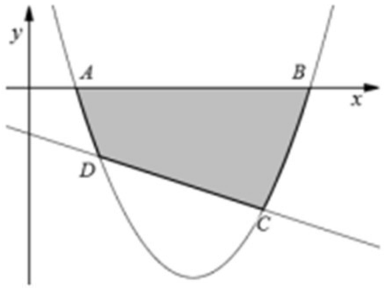
The coordinates of point C are $(8, 12)$.
The coordinates of point D are $(3, 7)$.

Find the exact value of the shaded area.

....

Question 12

The shaded shape $ABCD$ below is bounded by the x-axis, the curve $y = x^2 - 14x + 24$ and the line $y = -x - 6$.

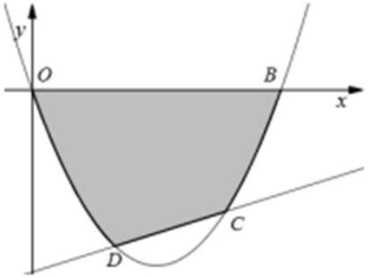


Find the exact value of the shaded area.

....

Question 13

The shaded shape $ABCD$ below is bounded by the x-axis, the curve $y = x^2 - 9x$ and the line passing through points C and D.



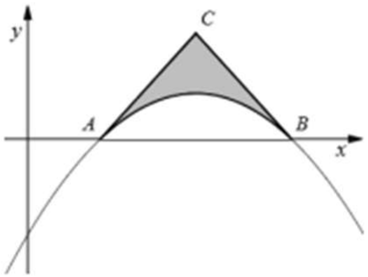
The coordinates of point C are $(7, -14)$.
The coordinates of point D are $(3, -18)$.

Find the exact value of the shaded area.

....

Question 14

The shaded shape below is formed by removing the area enclosed by the x-axis and the curve $y = -x^2 + 14x - 33$ from triangle ABC.



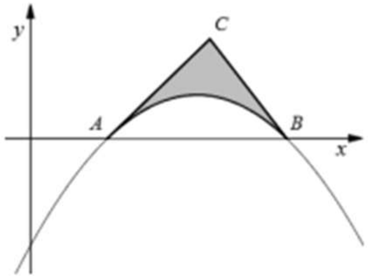
The coordinates of point C are $(7, 37)$.

Find the exact value of the shaded area.

....

Question 15

The shaded shape below is formed by removing the area enclosed by the x-axis and the curve $y = -x^2 + 13x - 30$ from triangle ABC.



The coordinates of point C are $(7, 28)$.

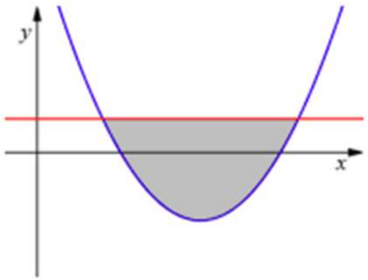
Find the exact value of the shaded area.

....

Question 16

The diagram shows the graphs of

$y = x^2 - 10x + 19$ and $y = 3$



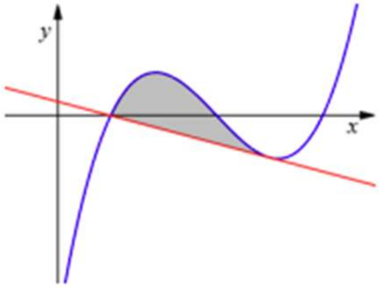
Use algebraic integration to find the exact value of the shaded area enclosed by the two graphs.

....

Question 17

The diagram shows the graphs of

$y = x^3 - 9x^2 + 23x - 15$ and $y = -x + 1$

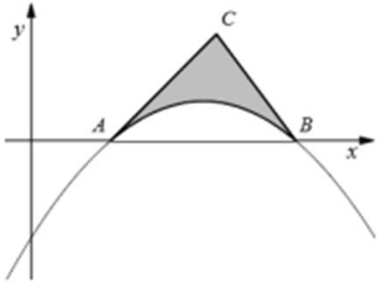


Use algebraic integration to find the exact value of the shaded area enclosed by the two graphs.

....

Question 18

The shaded shape below is formed by removing the area enclosed by the x-axis and the curve $y = -x^2 + 13x - 30$ from triangle ABC.



The coordinates of point C are (7,33).

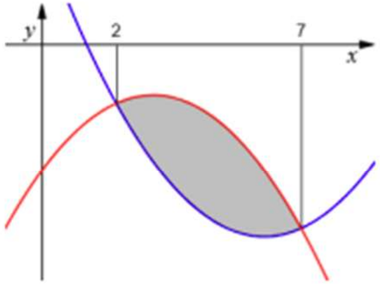
Find the exact value of the shaded area.

....

Question 19

The diagram shows the graphs of

$y = x^2 - 12x + 13$ and $y = -x^2 + 6x - 15$



The graphs intersect at the points where $x = 2$ and where $x = 7$.

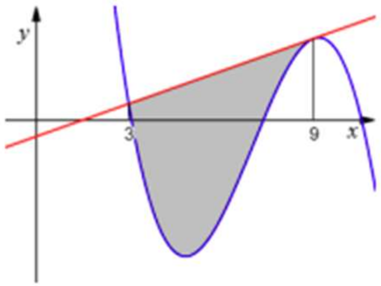
Use algebraic integration to find the exact value of the shaded area enclosed by the two graphs.

....

Question 20

The diagram shows the graphs of

$y = -x^3 + 21x^2 - 133x + 240$ and $y = 2x - 3$



The graphs intersect at the points where $x = 3$ and where $x = 9$.

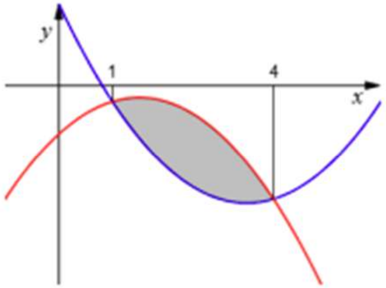
Use algebraic integration to find the exact value of the shaded area enclosed by the two graphs.

....

Question 21

The diagram shows the graphs of

$y = x^2 - 7x + 5$ and $y = -x^2 + 3x - 3$



The graphs intersect at the points where $x = 1$ and where $x = 4$.

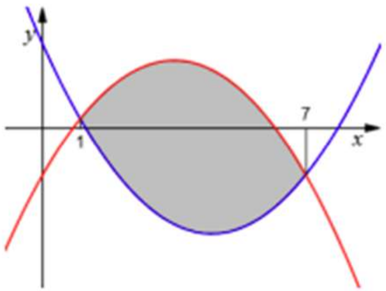
Use algebraic integration to find the exact value of the shaded area enclosed by the two graphs.

....

Question 22

The diagram shows the graphs of

$y = x^2 - 9x + 9$ and $y = -x^2 + 7x - 5$



The graphs intersect at the points where $x = 1$ and where $x = 7$.

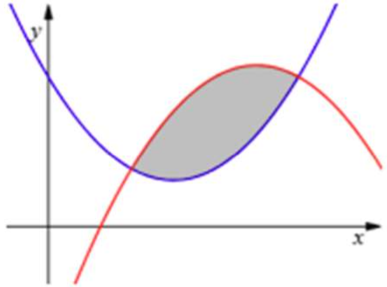
Use algebraic integration to find the exact value of the shaded area enclosed by the two graphs.

....

Question 23

The diagram shows the graphs of

$y = x^2 - 6x + 13$ and $y = -x^2 + 10x - 11$



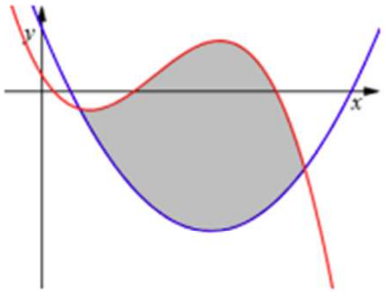
Find the exact value of the shaded area enclosed by the two graphs.

....

Question 24

The diagram shows the graphs of

$y = 3x^2 - 27x + 19$ and $y = -x^3 + 9x^2 - 18x + 5$



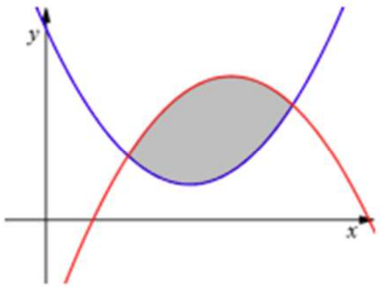
Find the exact value of the shaded area enclosed by the two graphs.

....

Question 25

The diagram shows the graphs of

$y = x^2 - 7x + 15$ and $y = -x^2 + 9x - 9$



Find the exact value of the shaded area enclosed by the two graphs.

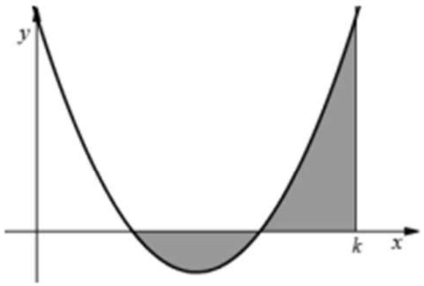
....

Question 26

The diagram below shows the graph of

$y = (x-3)(x-7)$

and the line $x = k$.



The total area of the shaded regions is $\frac{113}{3}$.

Find the value of k.

$k =$ ....

Question 27

You are given that

$$\int_1^k \left(\frac{2}{\sqrt{x}} + 5\right)\, dx = 87$$

where k is a positive constant.

Find the value of k.

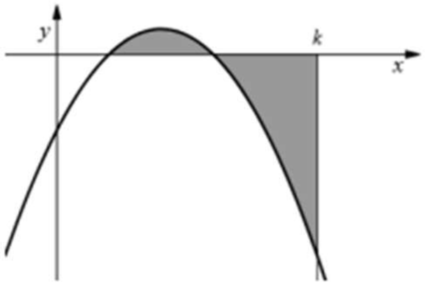
$k =$ ....

Question 28

The diagram below shows the graph of

$y = (x-1)(3-x)$

and the line $x = k$.



The total area of the shaded regions is 8.

Find the value of k.

$k =$ ....

Question 29

You are given that

$$\int_1^k \left(\frac{5}{\sqrt{x}} + 3\right) dx = 19$$

where k is a positive constant.

Find the value of k.

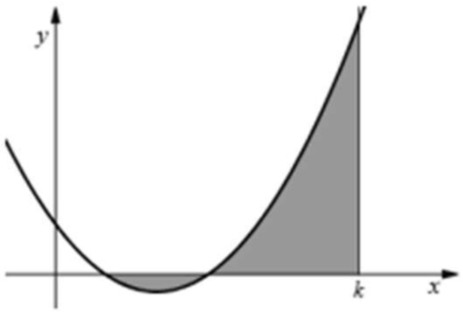
$k =$ ....

Question 30

The diagram below shows the graph of

$$y = (x-1)(x-3)$$

and the line $x = k$.



The total area of the shaded regions is $\frac{58}{3}$.

Find the value of k.

$k =$ ....

15.

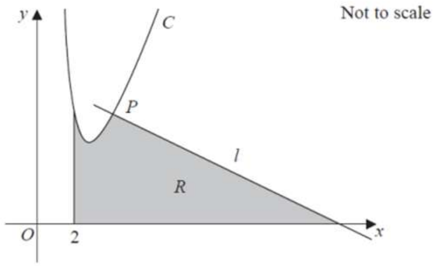


Figure 4

Figure 4 shows a sketch of part of the curve C with equation

$$y = \frac{32}{x^2} + 3x - 8, \qquad x > 0$$

The point P (4, 6) lies on C.
The line l is the normal to C at the point P.

The region R, shown shaded in Figure 4, is bounded by the line l, the curve C, the line with equation $x = 2$ and the x-axis.

Show that the area of R is 46
(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(10)

Your Turn

15.

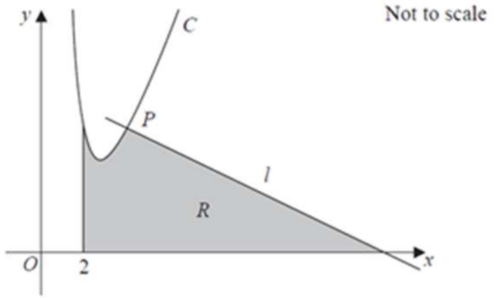


Figure 4

Figure 4 shows a sketch of part of the curve C with equation

$$y = \frac{32}{x^2} + 4x - 9, \quad x > 0.$$

The point P (4, 9) lies on C. The line l is the normal to C at the point P.

The region R, shown shaded in Figure 4, is bounded by the line l, the curve C, the line with equation $x = 2$ and the x-axis.

Show that the area of R is $\frac{271}{2}$.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(10)

14. A curve C has equation $y = f(x)$ where

$$f(x) = -3x^2 + 12x + 8$$

(a) Write f(x) in the form

$$a(x + b)^2 + c$$

where a, b and c are constants to be found.

The curve C has a maximum turning point at M.

(b) Find the coordinates of M.

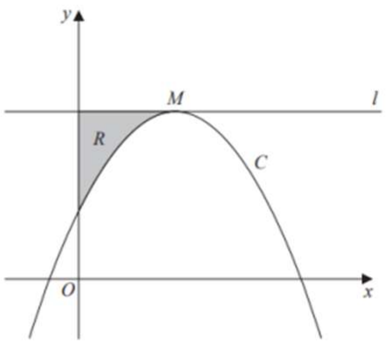


Figure 3

Figure 3 shows a sketch of the curve C.

The line l passes through M and is parallel to the x-axis.

The region R, shown shaded in Figure 3, is bounded by C, l and the y-axis.

(c) Using algebraic integration, find the area of R.

Your Turn

14. A curve C has equation $y = f(x)$ where

$$f(x) = -5x^2 + 30x + 6$$

(*a*) Write $f(x)$ in the form

$$a(x + b)^2 + c$$

where a, b and c are constants to be found.

(3)

The curve C has a maximum turning point at M.

(*b*) Find the coordinates of M.

(2)

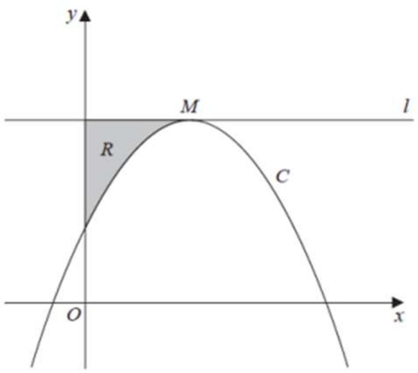


Figure 3

Figure 3 shows a sketch of the curve C.

The line l passes through M and is parallel to the x-axis.

The region R, shown shaded in Figure 3, is bounded by C, l and the y-axis.

(*c*) Using algebraic integration, find the area of R.

(5)

(Total for Question 14 is 10 marks)

10.

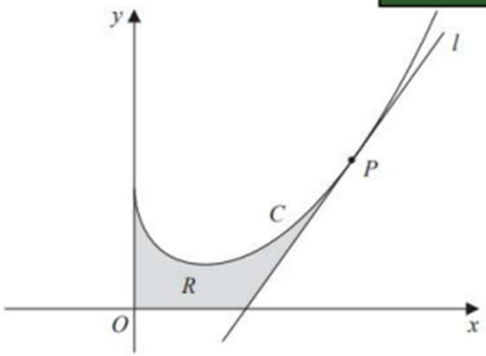


Figure 2

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Figure 2 shows a sketch of part of the curve C with equation

$$y = \frac{1}{3}x^2 - 2\sqrt{x} + 3 \qquad x \geqslant 0$$

The point P lies on C and has x coordinate 4

The line l is the tangent to C at P.

(a) Show that l has equation

$$13x - 6y - 26 = 0$$

(5)

The region R, shown shaded in Figure 2, is bounded by the y-axis, the curve C, the line l and the x-axis.

(b) Find the exact area of R.

(5)

Your Turn

10.

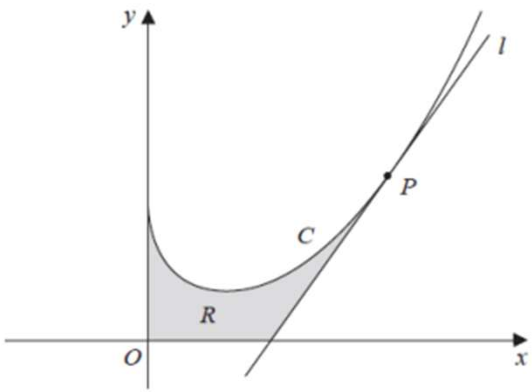


Figure 2

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Figure 2 shows a sketch of part of the curve C with equation

$$y = \frac{1}{4}x^2 - \sqrt{x} + 1 \qquad x \geq 0$$

The point P lies on C and has x coordinate 4

The line l is the tangent to C at P.

(a) Show that l has equation

$$7x - 4y - 16 = 0$$

(5)

The region R, shown shaded in Figure 2, is bounded by the y-axis, the curve C, the line l and the x-axis.

(b) Find the exact area of R.

(5)

(Total for Question 10 is 10 marks)

5. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

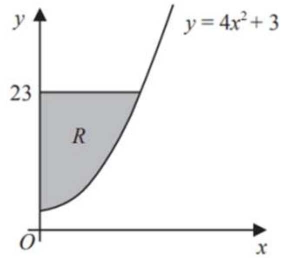


Figure 2

The finite region R, shown shaded in Figure 2, is bounded by the curve with equation $y = 4x^2 + 3$, the y-axis and the line with equation $y = 23$

Show that the exact area of R is $k\sqrt{5}$ where k is a rational constant to be found.

(5)

Your Turn

5. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

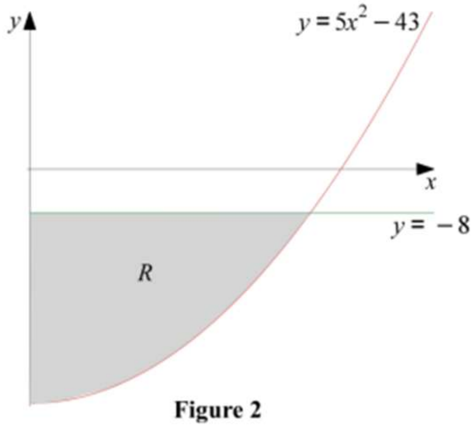


Figure 2

The finite region R, shown shaded in Figure 2, is bounded by the curve with equation $y = 5x^2 - 43$, the y-axis and the line with equation $y = -8$

Show that the exact area of R is $k\sqrt{7}$ where k is a rational constant to be found.

(5)

(Total for Question 5 is 5 marks)

AS 2019

Integration

3. (a) Given that k is a constant, find

$$\int\left(\frac{4}{x^3} + kx\right)\mathrm{d}x$$

simplifying your answer.

(3)

(b) Hence find the value of k such that

$$\int_{0.5}^{2}\left(\frac{4}{x^3} + kx\right)\mathrm{d}x = 8$$

(3)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$\int 4x^{-3} + kx \ \mathrm{d}x = -2x^{-2} + \frac{1}{2}kx$ [No Title]	M1	This mark is given for recognising that x^n becomes x^{n+1} when integrating
		A1	This mark is given for two correctly integrated terms (without c)
	$-\frac{2}{x^2} + \frac{kx^2}{2} + c$	A1	This mark is given for a full answer with a constant (in any correct form)
(b)	$\left[-\frac{2}{x^2} + \frac{k}{2}x^2\right]_{0.5}^{2} =$ $\left(-\frac{2}{2^2} + \frac{4k}{2}\right) - \left(-\frac{2}{0.5^2} + \frac{0.5^2 k}{2}\right) = 8$	M1	This mark is given for substituting the limits 2 and 0.5 and setting equal to 8
	$\left(-\frac{1}{2} + 2k\right) - \left(-8 + \frac{k}{8}\right) = 8$ $7.5 + \frac{15}{8}k = 8$	M1	This mark is given for a method to solve a linear equation in k
	$k = \frac{4}{15}$	A1	This mark is given for finding a correct value for k

Summary of Key Points

Summary of key points

1 If $\frac{dy}{dx} = x^n$, then $y = \frac{1}{n+1}x^{n+1} + c, n \neq -1$.

Using function notation, if $f'(x) = x^n$, then $f(x) = \frac{1}{n+1}x^{n+1} + c, n \neq -1$.

2 If $\frac{dy}{dx} = kx^n$, then $y = \frac{k}{n+1}x^{n+1} + c, n \neq -1$.

Using function notation, if $f'(x) = kx^n$, then $f(x) = \frac{k}{n+1}x^{n+1} + c, n \neq -1$.

When integrating polynomials, apply the rule of integration separately to each term.

3 $\int f'(x)dx = f(x) + c$

4 $\int(f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

5 To find the constant of integration, c

- Integrate the function
- Substitute the values (x, y) of a point on the curve, or the value of the function at a given point $f(x) = k$ into the integrated function
- Solve the equation to find c

6 If $f'(x)$ is the derivative of $f(x)$ for all values of x in the interval $[a, b]$, then the definite integral is defined as $\int_a^b f'(x)dx = [f(x)]_a^b = f(b) - f(a)$

7 The area between a positive curve, the x-axis and the lines $x = a$ and $x = b$ is given by

Area $= \int_a^b y\,dx$

where $y = f(x)$ is the equation of the curve.

8 When the area bounded by a curve and the x-axis is below the x-axis, $\int y\,dx$ gives a negative answer.

9 You can use definite integration together with areas of trapeziums and triangles to find more complicated areas on graphs.

T.306: Mixed Ex , P.011 : BSG Qs