KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Pure Mathematics
P2 5 Radians
Booklet

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

Prior knowledge check

1 Write down the exact values of the following trigonometric ratios.

a $\cos 120°$ **b** $\sin 225°$ **c** $\tan(-300°)$

d $\sin(-480°)$

← Year 1, Chapter 10

2 Simplify each of the following expressions.

a $(\tan\theta\cos\theta)^2 + \cos^2\theta$

b $1 - \dfrac{1}{\cos^2\theta}$ **c** $\sqrt{1 - \dfrac{\sin\theta\cos\theta}{\tan\theta}}$

← Year 1, Chapter 10

3 Show that

a $(\sin 2\theta + \cos 2\theta)^2 \equiv 1 + 2\sin 2\theta\cos 2\theta$

b $\dfrac{2}{\sin\theta} - 2\sin\theta \equiv \dfrac{2\cos^2\theta}{\sin\theta}$

← Year 1, Chapter 10

4 Solve the following equations for θ in the interval $0 \leqslant \theta \leqslant 360°$, giving your answers to 3 significant figures where they are not exact.

a $4\cos\theta + 2 = 3$ **b** $2\sin 2\theta = 1$

c $6\tan^2\theta + 10\tan\theta - 4 = \tan\theta$

d $10 + 5\cos\theta = 12\sin^2\theta$

← Year 1, Chapter 10

5.1 Radian Measure

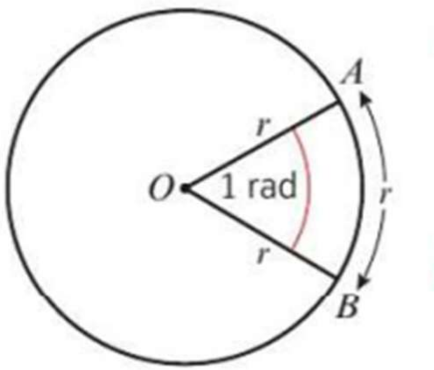
A degree is an *arbitrary* measure of an angle which comes from the idea that once there was believed to be 360 days in a year.

A radian on the other hand is not arbitrary as it represents that angle such that a single unit ensures a sectors arc length is equal to the radii.

Therefore, as circumference is $2\pi r$ it follows that

$$360° = 2\pi \text{ radians}$$

If the arc AB has length r, then $\angle AOB$ is 1 radian.



- **2π radians = 360°**
- **π radians = 180°**
- **1 radian = $\frac{180°}{\pi}$**

Notes

Fill in the blanks

Angle in Radians	Fraction of Circle	Angle in Degrees
2π	1	360
π		
		90
	$\frac{3}{4}$	
$\frac{\pi}{3}$		
		30
		120
$\frac{\pi}{4}$		
$\frac{4\pi}{3}$		
	$\frac{5}{8}$	

Angle in Radians	Fraction of Circle	Angle in Degrees
$\frac{7\pi}{4}$		
		330
$\frac{5\pi}{3}$		
	$\frac{5}{12}$	
		100
	$\frac{31}{36}$	
$\frac{8\pi}{15}$		
		306
θ		
		θ

Worked Example

Sketch the graph for $0 \le x \le 2\pi$ of $y = \sin x$

Worked Example

Sketch the graph for $0 \leq x \leq 2\pi$ of $y = \cos\ x$

Worked Example

Sketch the graph for $0 \leq x \leq 2\pi$ of $y = \tan\ x$

Worked Example

Sketch the graph for $0 \leq x \leq 2\pi$ of $y = \cos\left(x + \frac{\pi}{2}\right)$

Worked Example

Sketch the graph for $0 \leq x \leq 2\pi$ of $y = \cos(x + \pi)$

Worked Example

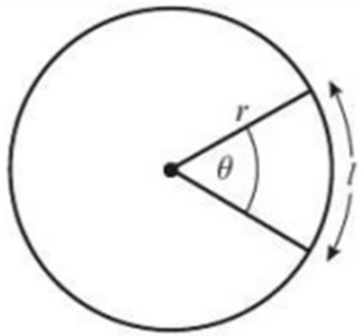
Sketch the graph for $0 \leq x \leq 2\pi$ of $y = \sin(2x)$

Ex 5A & 5B pgs 116 & 118

5.2 Arc Length

Using radians greatly simplifies the formula for **arc length**.

- **To find the arc length l of a sector of a circle use the formula $l = r\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.**



Notes

Worked Example

A triangle ABC is such that $AB = 8\ cm$, $AC = 11\ cm$ and $\angle BAC = 0.7$ radians.
The arc BD, where D lies on AC, is an arc of a circle with centre A and radius 8 cm.
A region R, is bounded by the straight lines BC and CD and the arc BD.
Find the perimeter of R

Worked Example

A sector of a circle of radius 15 cm contains an angle of θ radians. Given that the perimeter of the sector is 42 cm, find the value of θ

Worked Example

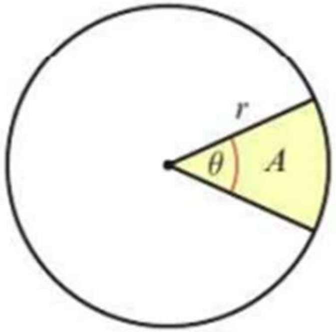
The perimeter of a sector OAB is four times the length of the arc AB. Find the size of angle AOB

Ex 5C pg 121 Qs 8+

5.3 Areas of Sectors and Segments

Using radians also greatly simplifies the formula for the area of a **sector**.

- **To find the area A of a sector of a circle use the formula $A = \frac{1}{2}r^2\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.**



Notes

Worked Example

A circle, centre O, radius 5.2 cm has a minor sector OAB where the arc AB subtends an angle of 0.8 radians at the centre of the circle.
A segment is enclosed by a chord AB and the arc AB.
Find the area of the segment.

Worked Example

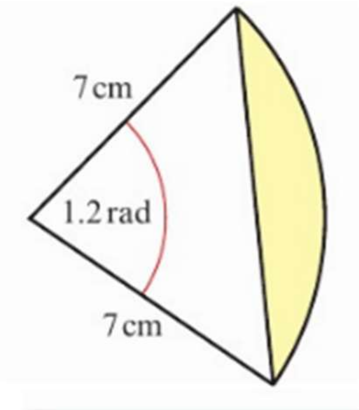
The area of the minor sector AOB is 28.9 cm^2. Given that $\angle AOB = 0.8$ radians and O is the centre of the circle, calculate the length of the radius

Worked Example

A sector of a circle of radius 55 m and perimeter 176 m.
Calculate the area of the sector

Worked Example

The diagram shows a sector of a circle. Find the area of the shaded segment.



Worked Example

OAB is a sector of a circle, centre O, radius $4m$.
The chord AB is 5m long.
Find the area of the segment.

Worked Example

AB is the diameter of a semicircle, centre O, radius r cm.
C is a point on the semicircle.
<BOC $= \theta$ radians.
Given that the area of ΔAOC is three times the segment enclosed by CB, show that $3\theta - 4\sin\theta = 0$

Worked Example

OAB is a sector of a circle, centre O, radius 9 cm and angle 0.7 radians.
C lies outside the sector.
AC is a straight line, perpendicular to OA.
OBC is a straight line.
Find the area of the region bounded by the arc AB and the lines AC and BC

Worked Example

OPQ is a sector of a circle, centre O, radius 10 cm where <POQ $= 0.3$ radians.
The point R is on OQ such that the ratio OR:RQ is $1:3$
A region is bounded by the arc PQ, QR and a line RP.

a) Find the perimeter of the region
b) Find the area of the region

Ex 5D pg 126 Qs 4+

3.

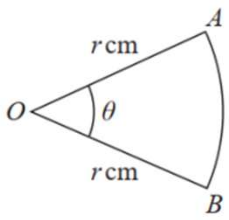


Figure 1

Figure 1 shows a sector AOB of a circle with centre O and radius r cm.

The angle AOB is θ radians.
The area of the sector AOB is 11 cm^2

Given that the perimeter of the sector is 4 times the length of the arc AB, find the exact value of r.

(4)

3.

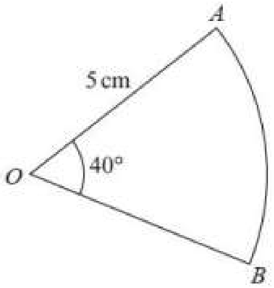


Figure 1

Figure 1 shows a sector AOB of a circle with centre O, radius 5 cm and angle $AOB = 40°$

The attempt of a student to find the area of the sector is shown below.

$$\text{Area of sector} = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2} \times 5^2 \times 40$$
$$= 500\,\text{cm}^2$$

(a) Explain the error made by this student.

(1)

(b) Write out a correct solution.

(2)

Your Turn

3.

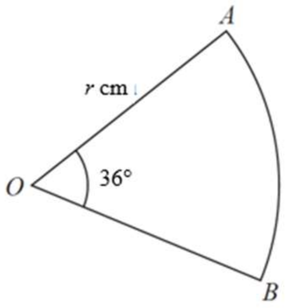


Figure 1

Figure 1 shows a sector *AOB* of a circle with centre *O*, radius $r = 7$ cm and angle $AOB = 36°$.

The attempt of a student to find the area of the sector is shown below.

$$\text{Area of sector} = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2} \times 7^2 \times 36$$
$$= 882 \text{ cm}^2$$

(*a*) Explain the error made by this student.

(1)

(*b*) Write out a correct solution.

(2)

(Total for Question 3 is 3 marks)

The shape *OABCDEFO* shown in Figure 1 is a design for a logo.

In the design
- *OAB* is a sector of a circle centre *O* and radius *r*
- sector *OFE* is congruent to sector *OAB*
- *ODC* is a sector of a circle centre *O* and radius 2*r*
- *AOF* is a straight line

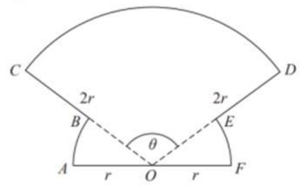


Figure 1

Given that the size of angle *COD* is θ radians,

(a) write down, in terms of θ, the size of angle *AOB*

(1)

(b) Show that the area of the logo is

$$\frac{1}{2}r^2(3\theta + \pi)$$

(2)

(c) Find the perimeter of the logo, giving your answer in simplest form in terms of r, θ and π.

(2)

6.

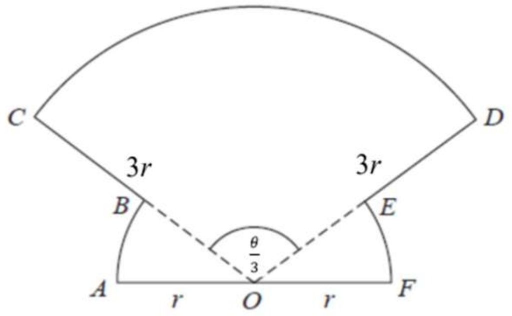


Figure 1

The shape *OABCDEFO* shown in Figure 1 is a design for a logo.

In the design
- *OAB* is a sector of a circle centre *O* and radius *r*
- sector *OFE* is congruent to sector *OAB*
- *ODC* is a sector of a circle centre *O* and radius 3*r*
- *AOF* is a straight line

Given that the size of angle *COD* is $\frac{\theta}{3}$ radians,

(*a*) write down, in terms of θ, the size of angle *AOB*

(1)

(*b*) Show that the area of the logo is

$$r^2\left(\frac{4}{3}\theta+\frac{\pi}{2}\right)$$

(2)

(*c*) Find the perimeter of the logo, giving your answer in simplest form in terms of r, θ and π.

(2)

(Total for Question 6 is 5 marks)

5.4 Solving Trigonometric Equations

Recap example:

Find, in degrees, the values of θ in the interval $0 \leqslant \theta \leqslant 360°$ for which

$2\cos^2\theta - \cos\theta - 1 = \sin^2\theta$

Give your answers to 1 decimal place, where appropriate.

Notes

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:

$\sin(\theta - \frac{\pi}{4}) = \frac{1}{2}$

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:

$\sin 3\theta = \frac{\sqrt{3}}{2}$

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:

$\sin^2 \theta = \frac{1}{4}$

Ex 5E pg 132 Qs 5-6

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:

$2\sin^2\theta - 5\sin\theta - 3 = 0$

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:
$5\sin^2\theta - 2\sin\theta = 0$

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:

$5\cos\theta\ sin\theta + 2\sin\theta = 0$

Worked Example

Solve the equation $17\cos\theta + 3\sin^2\theta = 13$ in the interval $0 \leq \theta \leq 2\pi$

Ex 5E pg 132 Qs 7+

5.5 Small Angle Approximations

You can use radians to find **approximations** for the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

- **When θ is small and measured in radians:**
  - $\sin\theta \approx \theta$
  - $\tan\theta \approx \theta$
  - $\cos\theta \approx 1 - \frac{\theta^2}{2}$

**these are given in formulae booklet*

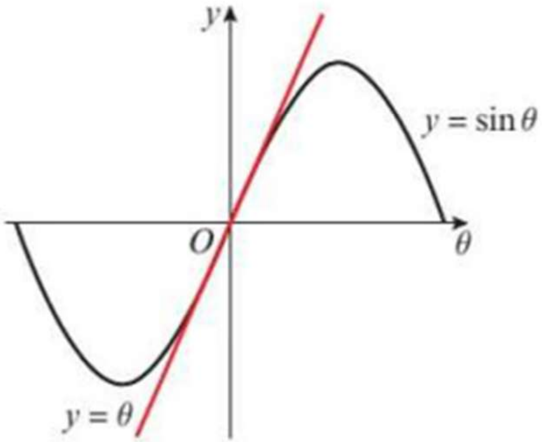
These are derived from series found in Further Maths (and the formulae booklet) where you only take the first few term(s)

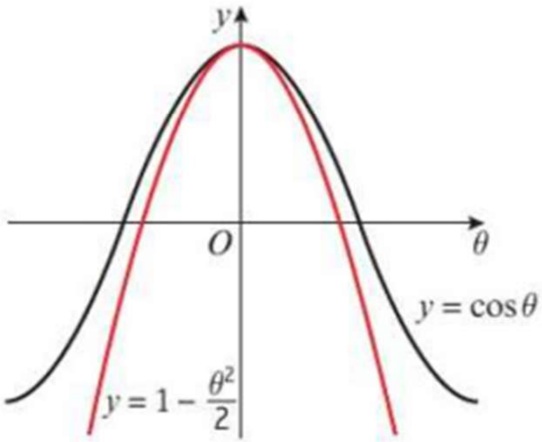
Maclaurin's and Taylor's Series

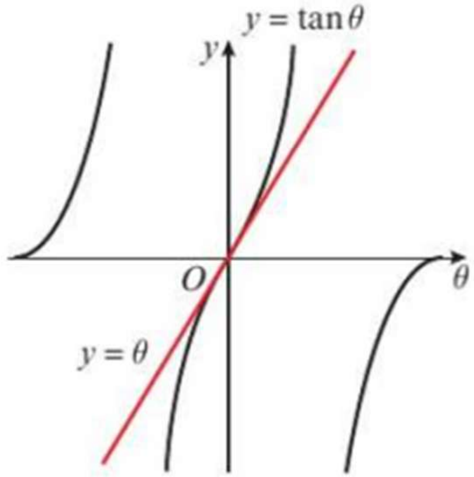
$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^r}{r!}f^{(r)}(0) + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \quad \text{for all } x$$







Notes

Worked Example

When θ is small, find the approximate value of:

a) $\frac{\sin 2\theta + \tan \theta}{2\theta}$

b) $\frac{\cos 4\theta - 1}{\theta \sin 2}$

c) $\sin 5\theta + \tan 2\theta - \cos 2\theta$

Worked Example

Find the percentage error when calculating the value of $\cos(0.246\ rad)$ using the small-angle approximations.

Worked Example

When θ is small, find the approximate value of:

$$\frac{1 - 2\tan\theta - 4\cos 2\theta}{\tan 2\theta + 1}$$

Ex 5F pg 137 Qs all

4. Given that θ is small and measured in radians, use the small angle approximations to show that

$$4\sin\frac{\theta}{2} + 3\cos^2\theta \approx a + b\theta + c\theta^2$$

where a, b and c are integers to be found.

(3)

Your Turn

4. Given that θ is small and measured in radians, use the small angle approximations to show that

$$40\sin\frac{\theta}{4} + 5\cos^2\theta \approx a + b\theta + c\theta^2$$

where a, b and c are integers to be found.

(3)

(Total for Question 4 is 3 marks)

4. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation $y = \mathrm{f}(x)$ where $x \in \mathbb{R}$

Given that

- $\mathrm{f}'(x) = 2x + \frac{1}{2}\cos x$
- the curve has a stationary point with x coordinate α
- α is small

(a) use the small angle approximation for $\cos x$ to estimate the value of α to 3 decimal places.

(3)

The point $P(0, 3)$ lies on C

(b) Find the equation of the tangent to the curve at P, giving your answer in the form $y = mx + c$, where m and c are constants to be found.

(2)

Your Turn

4. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation $y = \mathrm{f}(x)$ where $x \in \mathbb{R}$

Given that

- $\mathrm{f}'(x) = 3x - \frac{1}{3}\cos x$
- the curve has a stationary point with x coordinate α
- α is small

(a) use the small angle approximation for $\cos x$ to estimate the value of α to 3 decimal places.

(3)

The point $P(0, 5)$ lies on C

(b) Find the equation of the tangent to the curve at P, giving your answer in the form $y = mx + c$, where m and c are constants to be found.

(2)

Extract from Formulae book

Small angle approximations

$\sin\theta \approx \theta$

$\cos\theta \approx 1 - \frac{\theta^2}{2}$

$\tan\theta \approx \theta$

where θ is measured in radians

2.

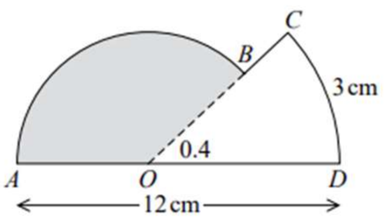


Figure 1

The shape *ABCDOA*, as shown in Figure 1, consists of a sector *COD* of a circle centre *O* joined to a sector *AOB* of a different circle, also centre *O*.

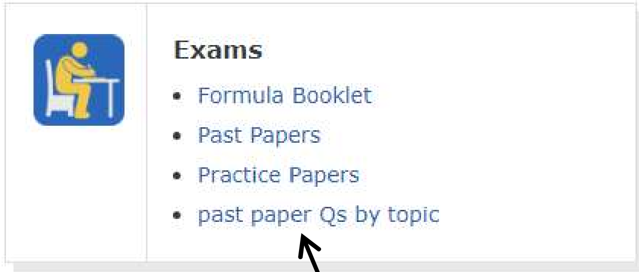
Given that arc length $CD = 3$ cm, $\angle COD = 0.4$ radians and *AOD* is a straight line of length 12 cm,

(a) find the length of *OD*,

(2)

(b) find the area of the shaded sector *AOB*.

(3)



Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

2(a)	Uses $s = r\theta \Rightarrow 3 = r \times 0.4$	M1	1.2
	$\Rightarrow OD = 7.5$ cm	A1	1.1b
		(2)	
(b)	Uses angle $AOB = (\pi - 0.4)$ or uses radius is $(12 - \text{"}7.5\text{"})$ cm	M1	3.1a
	Uses area of sector $= \frac{1}{2}r^2\theta = \frac{1}{2} \times (12 - 7.5)^2 \times (\pi - 0.4)$	M1	1.1b
	$= 27.8\text{cm}^2$	A1ft	1.1b
		(3)	
	(5 marks)		

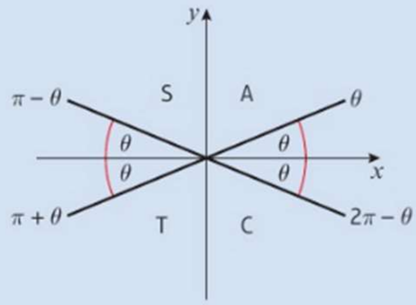
Summary of Key Points

1
- 2π radians = 360°
- π radians = 180°
- 1 radian = $\frac{180°}{\pi}$

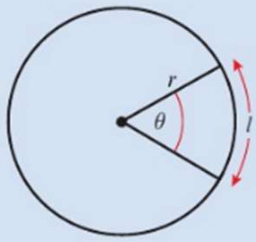
2
- $30° = \frac{\pi}{6}$ radians
- $45° = \frac{\pi}{4}$ radians
- $60° = \frac{\pi}{3}$ radians
- $90° = \frac{\pi}{2}$ radians
- $180° = \pi$ radians
- $360° = 2\pi$ radians

3 You need to learn the exact values of the trigonometric ratios of these angles measured in radians.
- $\sin\frac{\pi}{6} = \frac{1}{2}$
- $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
- $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
- $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
- $\cos\frac{\pi}{3} = \frac{1}{2}$
- $\tan\frac{\pi}{3} = \sqrt{3}$
- $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
- $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
- $\tan\frac{\pi}{4} = 1$

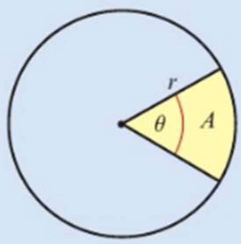
4 You can use these rules to find sin, cos or tan of any positive or negative angle measured in radians using the corresponding acute angle made with the x-axis, θ.
- $\sin(\pi - \theta) = \sin\theta$
- $\sin(\pi + \theta) = -\sin\theta$
- $\sin(2\pi - \theta) = -\sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\cos(\pi + \theta) = -\cos\theta$
- $\cos(2\pi - \theta) = \cos\theta$
- $\tan(\pi - \theta) = -\tan\theta$
- $\tan(\pi + \theta) = \tan\theta$
- $\tan(2\pi - \theta) = -\tan\theta$



5 To find the arc length l of a sector of a circle use the formula $l = r\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.



6 To find the area A of a sector of a circle use the formula $A = \frac{1}{2}r^2\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.



7 The area of a segment in a circle of radius r is

$$A = \frac{1}{2}r^2(\theta - \sin\theta)$$

8 When θ is small and measured in radians:
- $\sin\theta \approx \theta$
- $\tan\theta \approx \theta$
- $\cos\theta \approx 1 - \frac{\theta^2}{2}$

Mixed ex pg 135 and BSG P. 41