KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Pure Mathematics
8 The Binomial Expansion
Booklet

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

Prior knowledge check

1 Expand and simplify where possible:

a $(2x - 3y)^2$ **b** $(x - y)^3$ **c** $(2 + x)^3$

← Section 1.2

2 Simplify

a $(-2x)^3$ **b** $(3x)^{-4}$

c $\left(\frac{2}{5}x\right)^2$ **d** $\left(\frac{1}{3}x\right)^{-3}$

← Sections 1.1, 1.4

3 Simplify

a $(25x)^{\frac{1}{2}}$ **b** $(64x)^{-\frac{2}{3}}$

c $\left(\frac{9}{100}x\right)^{-\frac{1}{2}}$ **d** $\left(\frac{8}{27}x\right)^{\frac{4}{3}}$

← Section 1.4

8.3 The Binomial Expansion

Find the expansion of $(2+3x)^4$

Next have descending or ascending powers of one of the terms, going between 0 and 4 (note that if the power is 0, the term is 1, so we need not write it).

$$(2+3x)^4 = 1\,(2^4) + 4\,(2^3)(3x)^1 + 6\,(2^2)(3x)^2 + 4\,(2^1)(3x)^3 + 1\,(3x)^4$$

First fill in the correct row of Pascal's triangle.

And do the same with the second term but with powers going the opposite way, noting again that the 'power of 0' term does not appear.

Simplify each term (ensuring any number in a bracket is raised to the appropriate power)

$$= 16 + 96x + 216x^2 + 216x^3 + 81x^4$$

Notes

Worked Example

501i: Use a binomial expansion to reason about terms in the expansion of $(a+bx)^n(c+dx)$

The first three terms, in ascending powers of y, of the expansion of $(2y-1)^7$ are:

$$-1+14y-84y^2$$

Hence find the coefficient of y^2 in the expansion of $(4y+5)(2y-1)^7$.

Worked Example

501h: Expand $(a+bx)^n$ for positive integers n, where b is algebraic.

Find the first three terms, in ascending powers of x, of the binomial expansion of $(px+2)^9$.

Worked Example

Find the binomial expansion of $\left(x+\frac{1}{x}\right)^5$ giving each term in its simplest form.

T.164: 8C Q7, P.63: 8.3 Qs 7+

Exam Q

AS 2022

6. (a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of

$$\left(3 - \frac{2x}{9}\right)^8$$

giving each term in simplest form.

(4)

$$f(x) = \left(\frac{x-1}{2x}\right)\left(3 - \frac{2x}{9}\right)^8$$

(b) Find the coefficient of x^2 in the series expansion of f(x), giving your answer as a simplified fraction.

(2)

Your Turn

6. (*a*) Find the first 4 terms, in ascending powers of x, of the binomial expansion of

$$\left(2-\frac{5x}{6}\right)^7$$

giving each term in its simplest form.

$$f(x)=\left(\frac{x+1}{3x}\right)\left(2-\frac{5x}{6}\right)^7$$

(4)

(*b*) Find the coefficient of x^2 in the series expansion of f(x).

(2)

(Total for Question 6 is 6 marks)

8.4 Solving Binomial Problems

Factorial and Choose Function

$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$
said "n factorial", is the number of ways of arranging n objects in a line.

For example, suppose you had three letters, A, B and C, and wanted to arrange them in a line to form a 'word', e.g. ACB or BAC.

- There are 3 choices for the first letter.
- There are then 2 choices left for the second letter.
- There is then only 1 choice left for the last letter.

There are therefore $3 \times 2 \times 1 = 3! = 6$ possible combinations.
Your calculator can calculate a factorial using the $x!$ button.

$^nCr = \binom{n}{r} = \frac{n!}{r!(n-r)!}$
said "n choose r", is the number of ways of 'choosing' r things from n, such that the order in our selection does not matter.
These are also known as **binomial coefficients**.

For example, if you a football team captain and need to choose 4 people from amongst 10 in your class, there are $\binom{10}{4} = \frac{10!}{4!6!} = 210$ possible selections.
(Note: the $\binom{10}{4}$ notation is preferable to $^{10}C_4$)
Use the nCr button on your calculator (your calculator input should display "10C4")

Notes

Examples

Calculate the following <u>WITHOUT</u> a calculator

a) $\binom{5}{3}$

b) $\binom{n}{n-1}$

c) $\binom{n}{n-3}$

d) $\binom{2n}{n}$

e) $\binom{n}{1}$

f) $\binom{n}{n}$

Exercise

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

1. Simplify each of these:
   a. $\binom{n}{n-1}$
   b. $\binom{n}{3}$
   c. $\binom{n}{n-3}$
   d. $\binom{n+1}{n-1}$
   e. $\binom{n-2}{3}$
   f. $\binom{2n}{2}$
   g. $\binom{3n-1}{2}$
   h. $\binom{4n+5}{3}$
   i. $\binom{5n+4}{5n+2}$
   j. $\binom{2n-4}{2n-6}$

2. Solve the following equations:
   a. $3 \times {}^nC_6 = 11 \times {}^nC_4$
   b. $10 \times {}^nC_5 = 21 \times {}^nC_3$
   c. $33 \times {}^nC_2 = 2 \times {}^nC_5$
   d. $5 \times {}^nC_5 = 7 \times {}^nC_7$
   e. $28 \times {}^nC_4 = 15 \times {}^nC_6$
   f. $82 \times {}^nC_4 = 205 \times {}^nC_2$
   g. $7 \times {}^nC_7 = 117 \times {}^nC_5$
   h. $4 \times {}^nC_8 = 85 \times {}^nC_6$
   i. $5 \times {}^nC_6 = 506 \times {}^nC_3$
   j. $32509 \times {}^nC_4 = 35 \times {}^nC_7$

Worked Example

$g(x) = (1 + kx)^{10}$, where k is a constant.
Given that the coefficient of x^3 in the binomial expansion of $g(x)$ is 15, find the value of k.

T.166: 8D Q2,3, P.64: 8.4 Qs 1,2

Extra Exercise

3. In the binomial expansion of $(1+x)^n$, where $n \geq 5$, the coefficient of x^2 and the coefficient of x^5 is the same. Find the value of n.

4. In the binomial expansion of $(1+x)^n$, where $n \geq 4$, the coefficient of x^5 is 2 times the coefficient of x^4. Find the value of n.

5. In the binomial expansion of $(1+x)^n$, where $n \geq 4$, the coefficient of x^4 is 13 times the coefficient of x^2. Find the value of n.

6. In the binomial expansion of $(1+x)^n$, where $n \geq 6$, the coefficient of x^6 is 14 times the coefficient of x^2. Find the value of n.

7. In the binomial expansion of $(1+x)^n$, where $n \geq 4$, the coefficient of x^4 is the sum of 3 times the coefficient of x^3 and 4 times the coefficient of x^2. Find the value of n.

8. In the binomial expansion of $(1+x)^n$, where $n \geq 4$, the coefficient of x^3 is the sum of the coefficient of x^2 and 80 times the coefficient of x. Find the value of n.

9. In the binomial expansion of $(1+x)^n$, where $n \geq 7$, the coefficient of x^7 is the difference between 20 times the coefficient of x^4 and 17 times the coefficient of x^3. Find the value of n.

10. In the binomial expansion of $(1+x)^n$, where $n \geq 4$, the coefficient of x^4 is the difference between 2 times the coefficient of x^3 and 3 times the coefficient of x. Find the two possible values of n.

11. In the binomial expansion of $(1+2x)^n$, where $n \geq 2$, the coefficient of x^2 is 10 times the coefficient of x. Find the value of n.

Extra Exercise

12. In the binomial expansion of $(1+2x)^n$, where $n \geq 4$, the coefficient of x^4 is 70 times the coefficient of x. Find the value of n.

13. In the binomial expansion of $(1+2x)^n$, where $n \geq 5$, the coefficient of x^5 is the difference between 4400 times the coefficient of x and 1024. Find the value of n.

14. In the binomial expansion of $\left(1+\frac{x}{2}\right)^n$, where $n \geq 5$, the coefficient of x^4 is 2 times the coefficient of x^5. Find the value of n.

15. In the binomial expansion of $\left(1+\frac{x}{2}\right)^n$, where $n \geq 5$, the coefficient of x^4 is equal to the coefficient of x^5. Find the value of n.

[No Title]

16. In the binomial expansion of $(1+3x)^n$, where $n \geq 5$, the coefficient of x^5 is 1512 times the coefficient of x^2. Find the value of n.

17. In the binomial expansion of $(1+3x)^n$, where $n \geq 3$, the coefficient of x^3 is the sum of 6 times the coefficient of x^2 and 27 times the coefficient of x. Find the value of n.

18. In the binomial expansion of $(1+5x)^n$, where $n \geq 3$, the coefficient of x^3 is the difference between 32 times the coefficient of x^2 and 95 times the coefficient of x. Find the value of n.

Worked Example

a) Write down the first three terms, in ascending powers of x, of the binomial expansion of $(1+qx)^8$, where q is a non-zero constant.

b) Given that, in the expansion of $(1+qx)^8$, the coefficient of x is $-r$ and the coefficient of x^2 is $7r$, find the value of q and the value of r.

T.166: 8D Qs 3+, P.64: 8.4 Qs 3+

Exam Q

P2 2020

4. In the binomial expansion of

$$(a + 2x)^7 \qquad \text{where } a \text{ is a constant}$$

the coefficient of x^4 is 15 120

Find the value of a.

(3)

Your Turn

4. In the binomial expansion of

$$(a + 3x)^9 \qquad \text{where } a \text{ is a constant}$$

the coefficient of x^5 is 489 888.

Find the value of a.

(3)

(Total for Question 4 is 3 marks)

8.5 Binomial Estimation

Worked Example

a) Find the first three terms of the binomial expansion, in ascending powers of x, of $\left(7-\frac{x}{5}\right)^9$.

b) Use your expansion to estimate the value of 6.991^8, giving your answer to 4 significant figures.

T.168: 8E all Qs, P.65: 8.5 all Qs

Exam Q

AS 2019

8. (a) Find the first 3 terms, in ascending powers of x, of the binomial expansion of

$$\left(2 + \frac{3x}{4}\right)^6$$

giving each term in its simplest form.

(4)

(b) Explain how you could use your expansion to estimate the value of 1.925^6
You do not need to perform the calculation.

(1)

Your Turn

8.

(a) Find the first 3 terms, in ascending powers of x, of the binomial expansion of

$$\left(3 + \frac{4x}{5}\right)^5$$

(3)

(b) Explain how you could use your expansion to estimate the value of 2.96^5.
You do not need to perform this calculation

(2)

Extract from Formulae booklet

Binomial series

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \quad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

6. (a) Find the first 4 terms, in ascending powers of x, in the binomial expansion of

$$(1 + kx)^{10}$$

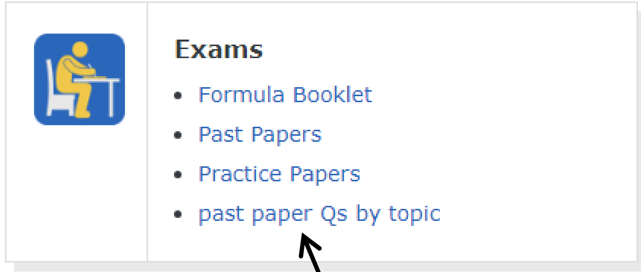
where k is a non-zero constant. Write each coefficient as simply as possible.

(3)

Given that in the expansion of $(1 + kx)^{10}$ the coefficient x^3 is 3 times the coefficient of x,

(b) find the possible values of k.

(3)



Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Question	Scheme	Marks	AOs
6 (a)	$(1+kx)^{10} = 1 + \binom{10}{1}(kx)^1 + \binom{10}{2}(kx)^2 + \binom{10}{3}(kx)^3 \ldots$	M1 A1	1.1b 1.1b
	$= 1 + 10kx + 45k^2x^2 + 120k^3x^3 \ldots$	A1	1.1b
		(3)	
(b)	Sets $120k^3 = 3 \times 10k$	B1	2.1
	$4k^2 = 1 \Rightarrow k = \ldots$	M1	1.1b
	$k = \pm\frac{1}{2}$	A1	1.1b
		(3)	
			(6 marks)

Summary of Key Points

1 Pascal's triangle is formed by adding adjacent pairs of numbers to find the numbers on the next row.

2 The $(n + 1)$th row of Pascal's triangle gives the coefficients in the expansion of $(a + b)^n$.

3 $n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$.

4 You can use factorial notation and your calculator to find entries in Pascal's triangle quickly.

- The number of ways of choosing r items from a group of n items is written as ${}^nC_r$ or $\binom{n}{r}$: ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$
- The rth entry in the nth row of Pascal's triangle is given by ${}^{n-1}C_{r-1} = \binom{n-1}{r-1}$.

5 The binomial expansion is:

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \quad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$

6 In the expansion of $(a + b)^n$ the general term is given by $\binom{n}{r}a^{n-r}b^r$.

7 If x is small, the first few terms in the binomial expansion can be used to find an approximate value for a complicated expression.