KING EDWARD VI HANDSWORTH GRAMMAR SCHOOL FOR BOYS

KING EDWARD VI ACADEMY TRUST BIRMINGHAM

Year 12
Statistics 2
Chapter 2 – Conditional Probability

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior Knowledge Check

1 Events A and B are mutually exclusive.
P(A) = 0.3 and P(B) = 0.45. Find:

a P(A or B)

b P(A and B)

c P(neither A nor B). ← Year 1, Chapter 5

2 Events C and D are independent.
P(C) = 0.2 and P(D) = 0.6.

a Find P(C and D).

b Draw a Venn diagram to show events C and D and the whole sample space.

c Find P(neither C nor D). ← Year 1, Chapter 5

3 A bag contains seven counters numbered 1–7. Two counters are selected at random without replacement. Find the probability that:

a Both counters are odd-numbered

b At least one counter is odd-numbered.

← Year 1, Chapter 5

2.1 Set Notation

Recap – Using Sets for Sample Spaces and Events

In general, sets are used to represent **collections of items**.

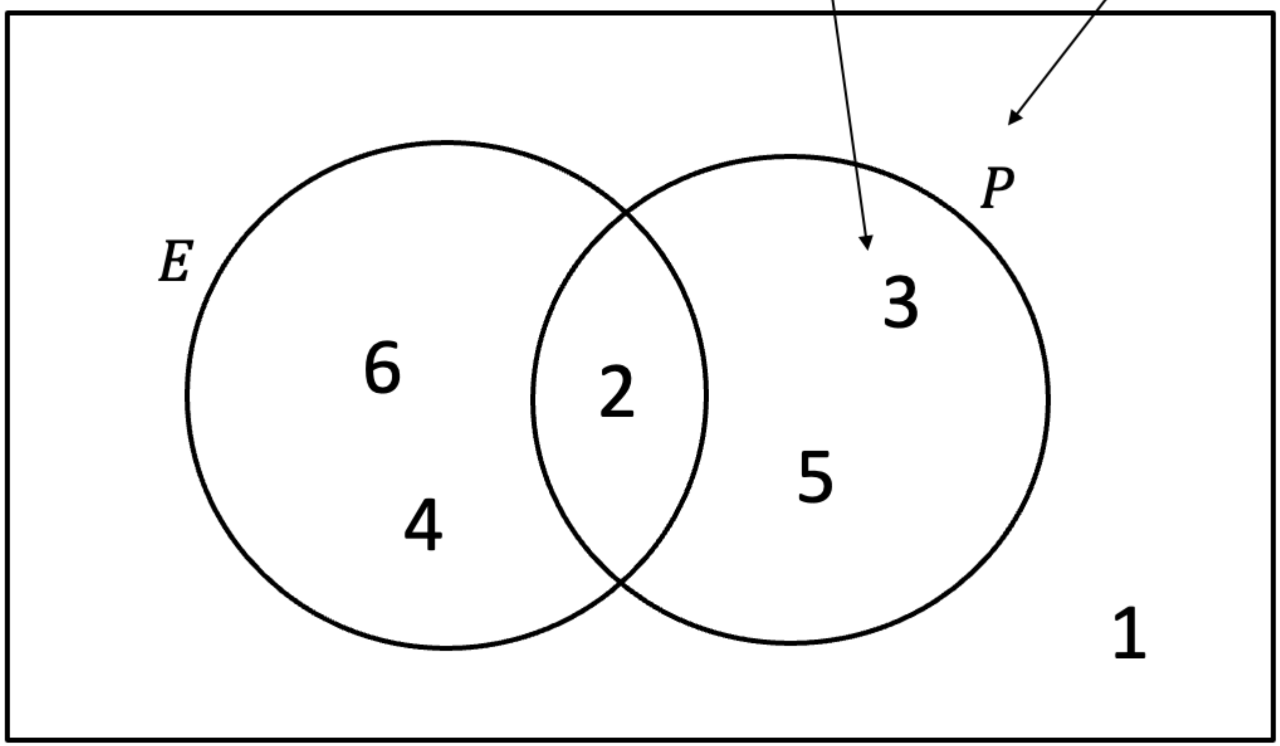
A **sample space** is **set of all possible outcomes**. We use ξ (Greek 'Xi'), or sometimes just S, to represent this set. We use a rectangle in a Venn Diagram.

Each number represents an **outcome**.

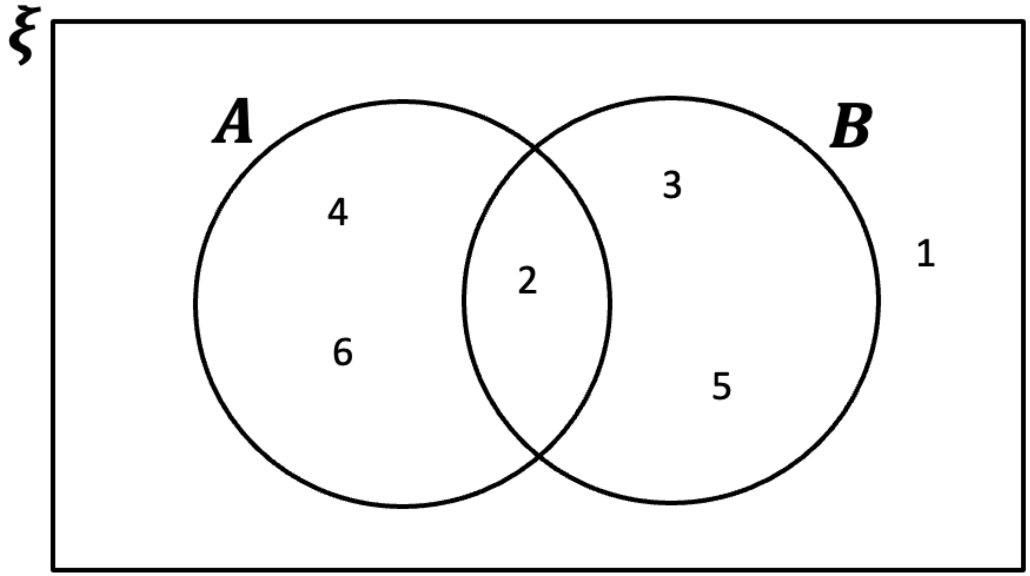
In probability, an **event** is a **set of one or more outcomes**. These are the circles in the Venn Diagram. We use capital letters for the variables representing sets.

$E =$ "rolling an even number"
$P =$ "rolling a prime number"

ξ



Combining Events and Sets



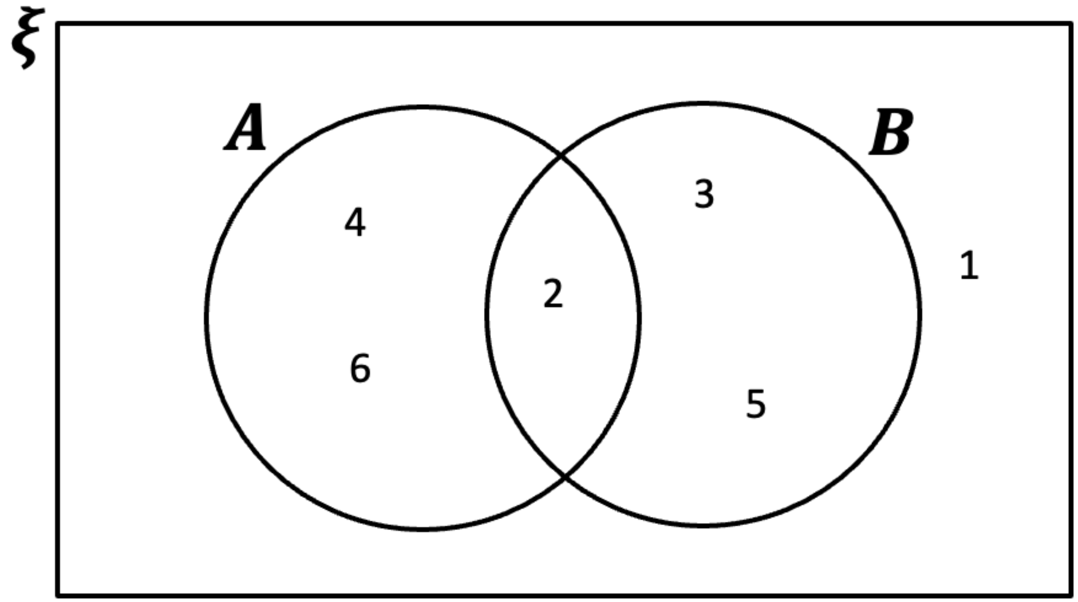
ξ = the whole sample space (1 to 6)

A = even number on a die thrown

B = prime number on a die thrown

	What does it mean in this context?	What is the resulting set of outcomes?
A'	**Not A (the "<u>complement</u>" of A).** i.e. Not rolling an even number.	$\{1, 3, 5\}$
$A \cup B$	**A or B (the "<u>union</u>" of A and B).** i.e. Rolling an even or prime number.	$\{2,3,4,5,6\}$
$A \cap B$	**A and B (the "<u>intersection</u>" of A and B).** i.e. Rolling a number which is even and prime.	$\{2\}$

Some Fundamentals



S = the whole sample space (1 to 6)

A = even number on a die thrown

B = prime number on a die thrown

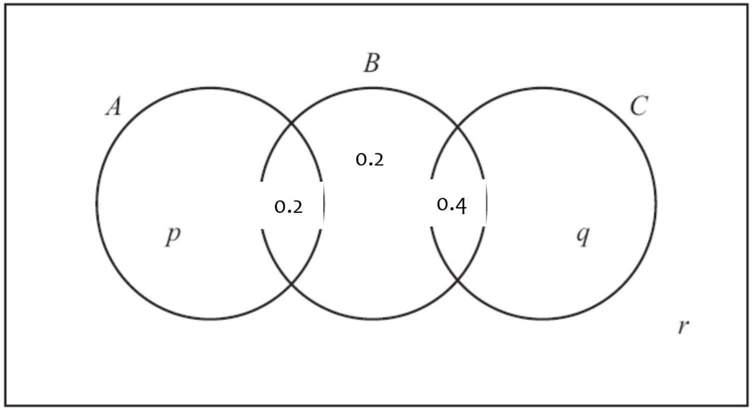
	What does it mean in this context?	What is the resulting set of outcomes?
$A \cap B'$	**"A and not B".** Rolling a number which is even and not prime.	$\{4,6\}$
$(A \cup B)'$	Rolling a number which is not [even or prime].	$\{1\}$
$(A \cap B)'$	Rolling a number which is not [even and prime].	$\{1,3,4,5,6\}$

Notes

Worked Example

The events A and B are independent.

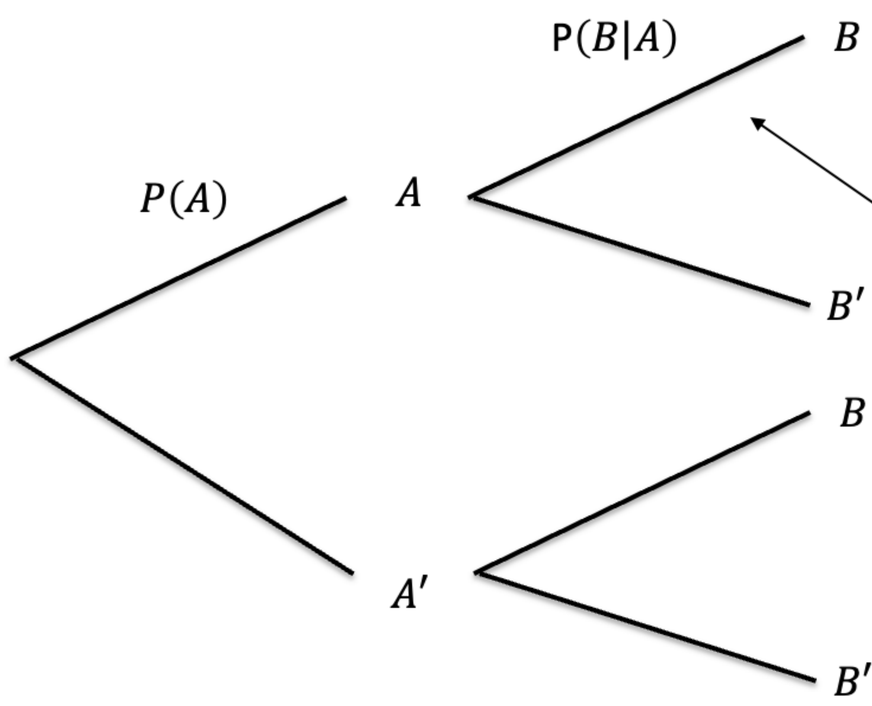
Find the value of p



2.2 Conditional Probability

Conditional Probability

Think about how we formed a probability tree at GCSE:



$P(A \cap B)$
$= P(A) \times P(B|A)$

Read the '|' symbol as "**given that**". i.e. "B occurred **given that** A occurred".

Alternatively (and more commonly):

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Memory Tip: You're dividing by the event you're conditioning on.

Notes

2.3 Conditional Probabilities in Venn Diagrams

What is Meant by a Conditional Probability?

In a lottery you pick a number from 1 to 1000. A number is chosen at random from these, and you win if the number matches.
What is the probability that:

a You flip a Heads on a fair coin **and** win the lottery.

$$\frac{1}{2} \times \frac{1}{1000} = \frac{1}{2000}$$

Winning the lottery and flipping a Heads are **independent** of each other, thus we can multiply the probabilities.

b You flip a Heads on a fair coin **given that** you won the lottery.

$$\frac{1}{2}$$

This is a trick question! The win on the lottery has **already happened**, so we needn't consider the probability of it. You're in the scenario where you've won the lottery, and now considering the act of flipping a coin, which is independent of having won the lottery.

Conditional Probability Notation

"given that"

$$P(A|B)$$

This means the probability that A occurred, given that B occurred.

We say B is the event being conditioned on, i.e. the context in which we are considering the probability of A.

Notes

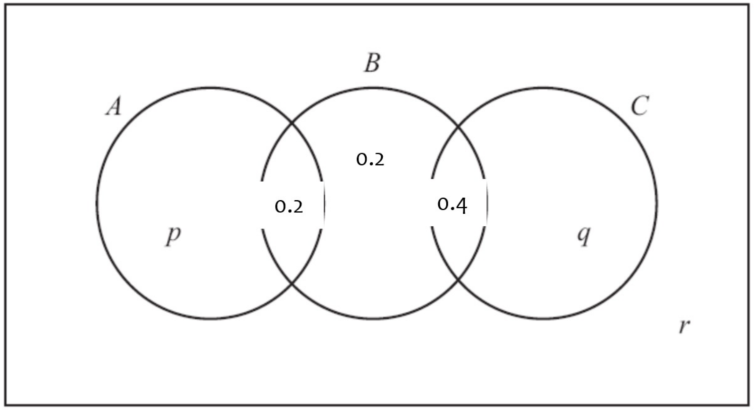
Worked Example

The events A and B are independent.

$P(B|C) = \frac{10}{11}$ and $p = 0.05$

a) Find the values of q and r

b) Find $P(A \cup C|B)$



2.4 Probability Formulae

The Addition Rule

Suppose that we knew that:

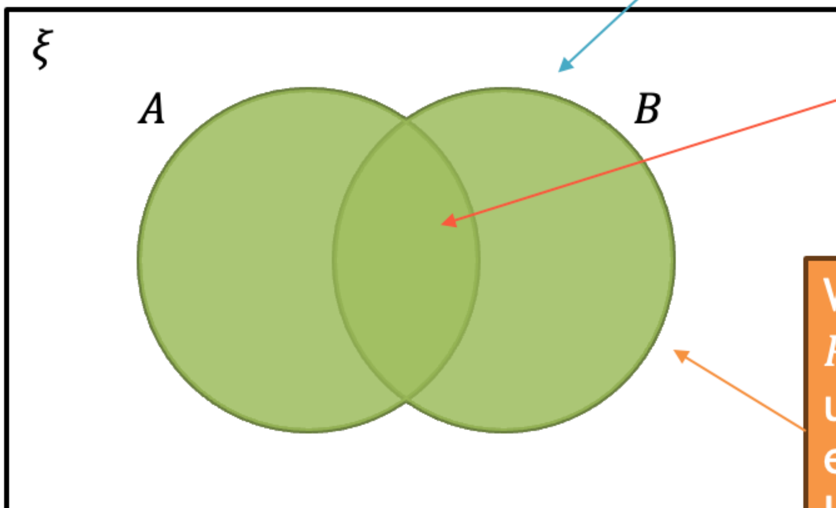
$$P(A) = 0.4$$
$$P(B) = 0.7$$
$$P(A \cap B) = 0.15$$

How could we work out $P(A \cup B)$?

These 3 regions make up $P(A \cup B)$, but we only want to include the probability in each region <u>**once**</u>.

If we subtract the intersection once, then each of the 3 regions will be included exactly once

We can first find $P(A) + P(B)$, i.e. adding up the probabilities in each of the two circles. However, this <u>**double counts the intersection**</u>.

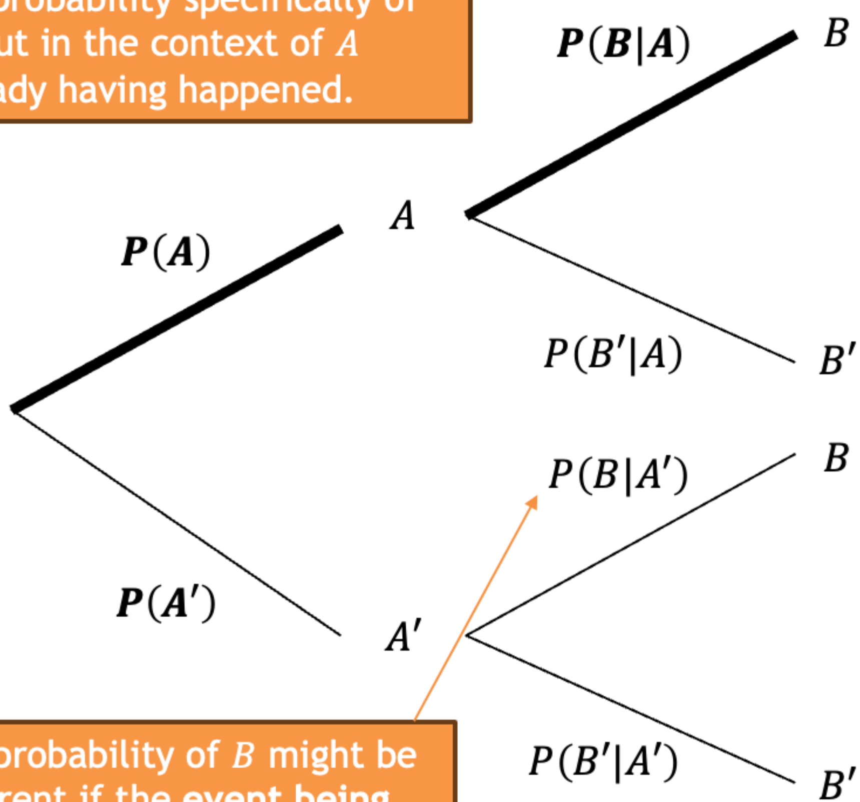


The Addition Rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Understanding Conditional Events on Trees

$P(B|A)$ means the probability that B happened "given that" A happened. So it's considering the probability specifically of B, but in the context of A already having happened.



→ $P(A \cap B)$

We have previously seen that to determine the probability of a <u>**sequence of events**</u>, we multiply them together, i.e.

- A occurred then
- B occurred, <u>**given that**</u> A occurred.

Thus:

$$P(A) \times P(B|A) = P(A \cap B)$$

Rearranging:

The probability of B might be different if the <u>**event being conditioned on**</u> (in this case A not happening) has changed.

Conditional probability formula:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Full Laws of Probability

If events A and B are <u>independent</u>.

$$P(A \cap B) = P(A) \times P(B)$$
$$P(A|B) = P(A)$$

If events A and B are <u>mutually exclusive</u>:

$$P(A \cap B) = 0$$
$$P(A \cup B) = P(A) + P(B)$$

In general:

We first encountered this in the previous section.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

This is known as the **<u>Addition Law</u>.** **Informal Proof**: If we added the probabilities in the A and B sets in the Venn Diagram, we'd be double counting the intersection, so subtract so that it's only counted once.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Super Important Tips

If I were to identify two tips that will possible help you the most in probability questions:

If you see the words '*given that*', Immediately write out the law for conditional probability.

Example: "Given Bob walks to school, find the probability that he's not late…"

First thing you should write: $P(L'|W) = \frac{P(L' \cap W)}{P(W)} = \cdots$

If you see the words '*are independent*', Immediately write out the laws for independence.
(Even before you've finished reading the question!)

Example: "A is independent from B…"

First thing you should write:

$$P(A \cap B) = P(A)P(B)$$
$$P(A|B) = P(A)$$

If you're stuck on a question where you have to find a probability given others, it's probably because you've failed to take into account that two events are independent or mutually exclusive, or you need to use the conditional probability or additional law.

Notes

Worked Example

C and D are two independent events such that

$P(C) = \frac{1}{3}$

$P(C \cup D) = \frac{3}{5}$

Find:

a) $P(D)$

b) $P(C' \cap D)$

c) $P(D'|C)$

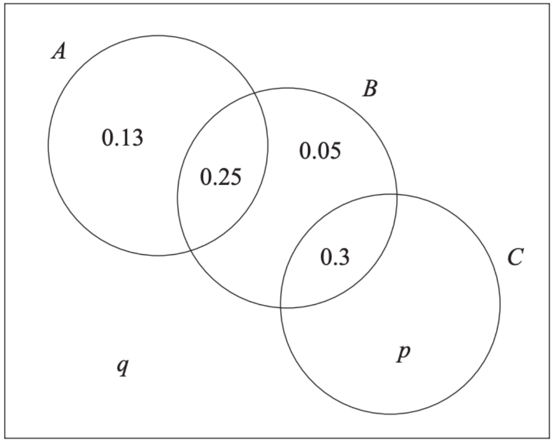
Worked Example

There are three events: A, B and C.
A and C are mutually exclusive.
A and B are independent.
$P(A) = 0.4$
$P(C) = 0.3$
$P(A \cup B) = 0.6$

Find:
a) $P(A|B)$
b) $P(A \cup C)$
c) $P(B)$

Worked Example

The Venn diagram, where p and q are probabilities, shows the three events A, B and C and their associated probabilities.



(a) Find P(A)

(1)

The events B and C are independent.

(b) Find the value of p and the value of q

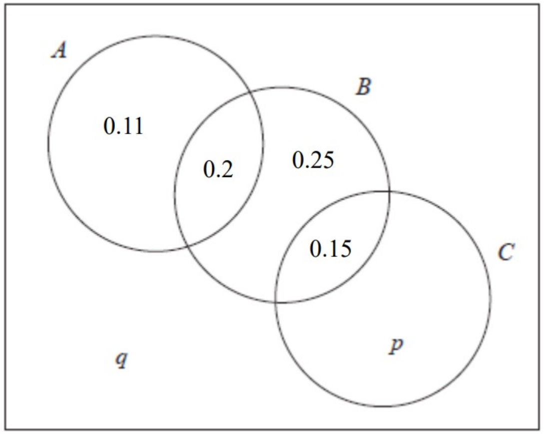
(3)

(c) Find $P(A|B')$

(2)

Your Turn

The Venn diagram, where p and q are probabilities, shows the three events A, B and C and their associated probabilities.



(a) Find P(A)

(1)

The events B and C are independent.

(b) Find the value of p and the value of q

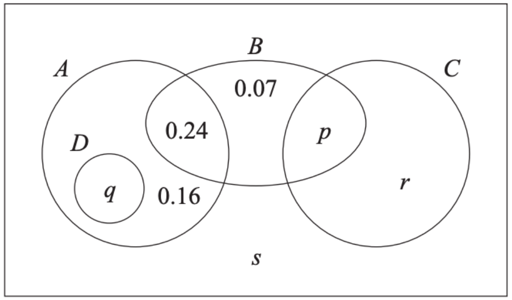
(3)

(c) Find $P(B|A')$

(2)

Worked Example

The Venn diagram shows the probabilities associated with four events, A, B, C and D



(a) Write down any pair of mutually exclusive events from A, B, C and D

(1)

Given that $P(B) = 0.4$

(b) find the value of p

(1)

Given also that A and B are independent

(c) find the value of q

(2)

Given further that $P(B' \mid C) = 0.64$

(d) find

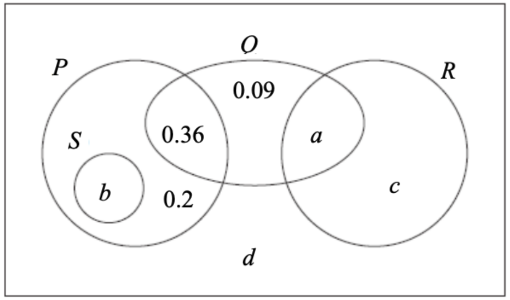
(i) the value of r

(ii) the value of s

(4)

Your Turn

The Venn diagram shows the probabilities associated with four events, P, Q, R and S



(a) Write down any pair of mutually exclusive events from P, Q, R and S

(1)

Given that P(Q) = 0.5

(b) find the value of a

(1)

Given also that P and Q are independent

(c) find the value of b

(2)

Given further that $P(Q' \mid R) = 0.6$

(d) find

(i) the value of c

(ii) the value of d

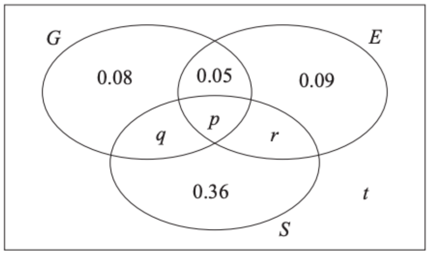
(4)

Worked Example

A large college produces three magazines.
One magazine is about green issues, one is about equality and one is about sports.
A student at the college is selected at random and the events G, E and S are defined as follows

G is the event that the student reads the magazine about green issues
E is the event that the student reads the magazine about equality
S is the event that the student reads the magazine about sports

The Venn diagram, where p, q, r and t are probabilities, gives the probability for each subset.



(a) Find the proportion of students in the college who read exactly one of these magazines.

(1)

No students read all three magazines and $P(G) = 0.25$

(b) Find

(i) the value of p

(ii) the value of q

(3)

Given that $P(S \mid E) = \frac{5}{12}$

(c) find

(i) the value of r

(ii) the value of t

(4)

(d) Determine whether or not the events $(S \cap E')$ and G are independent.
Show your working clearly.

(3)

Your Turn

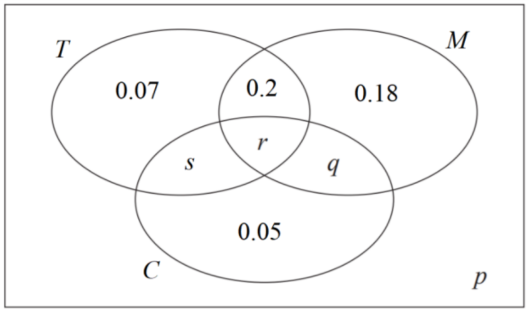
The Venn diagram shows the results from a random sample of people.

T is the event that the person drinks tea

M is the event that the person drinks milk

C is the event the person drinks coffee

The Venn diagram, where p, q, r and s are probabilities, gives the probability for each subset.



(*a*) Find the proportion of people who drink exactly one of the drinks.

(1)

No person likes all three drinks and $P(T) = 0.41$

(*b*) Find

(i) the value of r

(ii) the value of s

(3)

Given that $P(M \mid C) = \frac{30}{49}$

(*c*) find

(i) the value of q

(ii) the value of p

(4)

(*d*) Determine whether or not the events $(T \cap M')$ and C are independent.
Show your working clearly.

(3)

Worked Example

A company has 1825 employees.
The employees are classified as professional, skilled or elementary.

The following table shows

- the number of employees in each classification
- the two areas, A or B, where the employees live

	A	***B***
Professional	740	380
Skilled	275	90
Elementary	260	80

An employee is chosen at random.

Find the probability that this employee

(a) is skilled,

(1)

(b) lives in area B and is not a professional.

(1)

Some classifications of employees are more likely to work from home.

- 65% of professional employees in both area A and area B work from home
- 40% of skilled employees in both area A and area B work from home
- 5% of elementary employees in both area A and area B work from home
- Event F is that the employee is a professional
- Event H is that the employee works from home
- Event R is that the employee is from area A

(c) Using this information, complete the Venn diagram on the opposite page.

(4)

(d) Find $P(R' \cap F)$

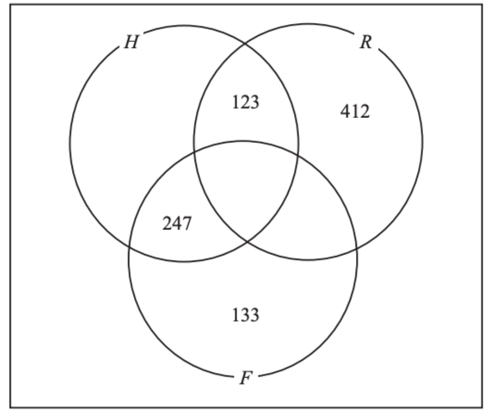
(1)

(e) Find $P([H \cup R]')$

(1)

(f) Find $P(F \mid H)$

(2)



Turn over for a spare diagram if you need to redraw your Venn diagram.

Your Turn

A company has 1670 employees.
The employees are classified as professional, skilled or elementary.

The following table shows

- the number of employees in each classification
- the two areas, A or B, where the employees live

	A	B
Professional	625	345
Skilled	270	120
Elementary	250	60

An employee is chosen at random.

Find the probability that this employee

(a) is skilled,

(1)

(b) lives in area B and is not a professional.

(1)

Some classifications of employees are more likely to work from home.

- 60% of professional employees in both area A and area B work from home
- 30% of skilled employees in both area A and area B work from home
- 10% of elementary employees in both area A and area B work from home
- Event F is that the employee is a professional
- Event H is that the employee works from home
- Event R is that the employee is from area A

(c) Using this information, complete the Venn diagram.

(4)

(d) Find $\mathrm{P}(R' \cap F)$

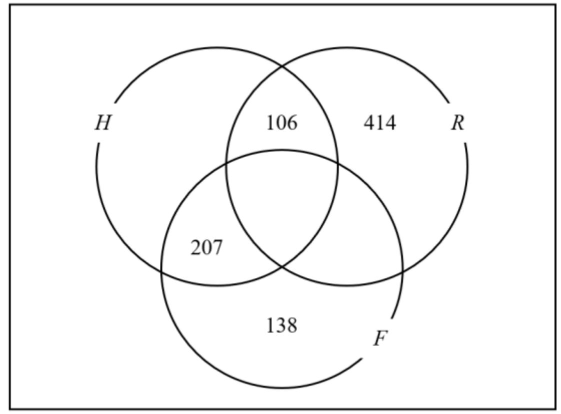
(1)

(e) Find $\mathrm{P}([H \cup R]')$

(1)

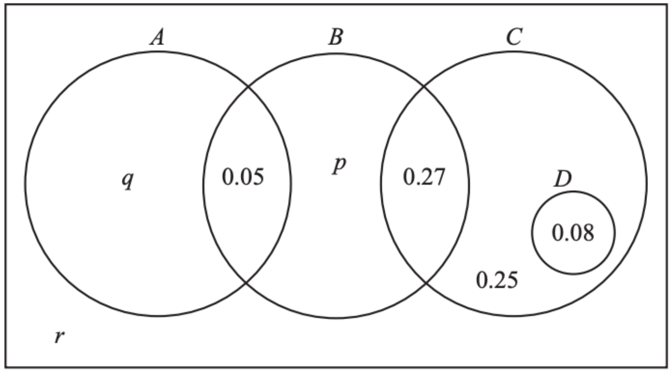
(f) Find $\mathrm{P}(F \mid H)$

(2)



Worked Example

The Venn diagram, where p, q and r are probabilities, shows the events A, B, C and D and associated probabilities.



(a) State any pair of mutually exclusive events from A, B, C and D

(1)

The events B and C are independent.

(b) Find the value of p

(2)

(c) Find the greatest possible value of $P(A \mid B')$

(3)

Given that $P(B \mid A') = 0.5$

(d) find the value of q and the value of r

(3)

(e) Find $P\left([A \cup B]' \cap C\right)$

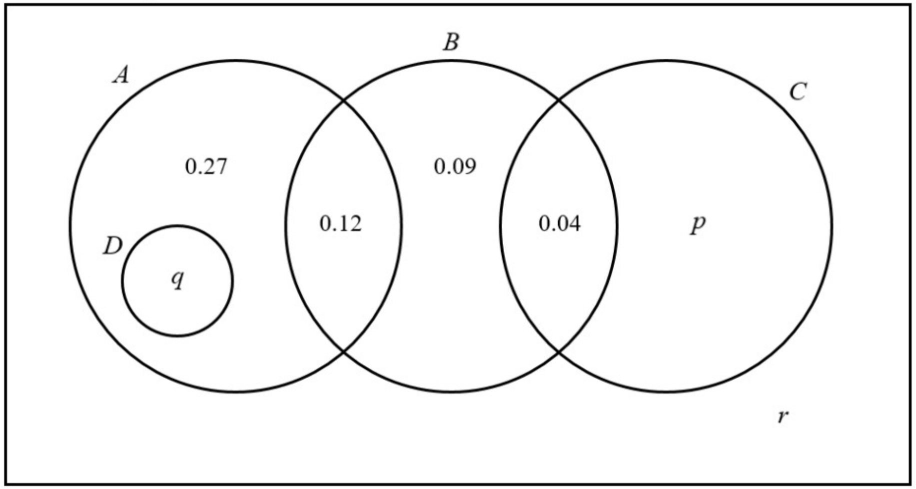
(1)

(f) Use set notation to write an expression for the event with probability p

(1)

Your Turn

The Venn diagram, where p, q and r are probabilities, shows the events A, B, C and D and associated probabilities.



(a) State any pair of mutually exclusive events from A, B, C and D

(1)

The events A and B are independent.

(b) Find the value of q

(2)

(c) Find the greatest possible value of $P(C|B')$

(3)

Given that $P(B|C) = 0.25$

(d) find the value of p and the value of r

(3)

(e) Find $P([B \cup C]' \cap A)$

(1)

(f) Use set notation to write an expression for an event not containing D, with a probability equal to $P(D)$

(1)

2.5 Tree Diagrams

Notes

Worked Example

A person plays a game of tennis and then a game of golf.
They can only win or lose each game.
The probability of winning tennis is 0.3
The probability of winning golf is 0.7
The results of each game are independent of each other.
Calculate the probability that the person wins at least one game.

Worked Example

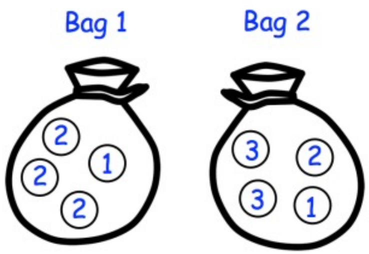
The table shows 100 students, who each study one language.
Two students are chosen at random.

	French	German
Female	26	30
Male	10	34

Calculate the probability that the two chosen students study the same language.

Worked Example

There are two bags with numbered discs as shown.



A person chooses a disc at random from bag 1.
If it is labelled 1, he puts the disc in bag 2.
If it is labelled 2, he does not put the disc in bag 2.
He then chooses a disc at random from bag 2.
He then adds the numbers of the two discs he selected to give his score.
Find the probability that his score is 4.

Worked Example

Three bags, A, B and C, each contain 1 red marble and some green marbles.

Bag A contains 1 red marble and 9 green marbles only
Bag B contains 1 red marble and 4 green marbles only
Bag C contains 1 red marble and 2 green marbles only

Sasha selects at random one marble from bag A.
If he selects a red marble, he stops selecting.
If the marble is green, he continues by selecting at random one marble from bag B.
If he selects a red marble, he stops selecting.
If the marble is green, he continues by selecting at random one marble from bag C.

(a) Draw a tree diagram to represent this information.
(2)

(b) Find the probability that Sasha selects 3 green marbles.
(2)

(c) Find the probability that Sasha selects at least 1 marble of each colour.
(2)

(d) Given that Sasha selects a red marble, find the probability that he selects it from bag B.
(2)

Your Turn

Three boxes, A, B and C, each contain 1 red scarf and some green scarves.

Box A contains 1 red scarf and 7 green scarves only
Box B contains 1 red scarf and 7 green scarves only
Box C contains 1 red scarf and 9 green scarves only

Julia selects at random one scarf from Box A.
If she selects a red scarf, she stops selecting.
If the scarf is green, she continues by selecting at random one scarf from Box B.
If she selects a red scarf, she stops selecting.
If the scarf is green, she continues by selecting at random one scarf from Box C.

(a) Draw a tree diagram to represent this information.

(2)

(b) Find the probability that Julia selects 3 green scarves.

(2)

(c) Find the probability that Julia selects at least 1 scarf of each colour.

(2)

(d) Given that Julia selects a red scarf, find the probability that she selects it from Box B.

(2)

Summary

1 The event A and B can be written as $A \cap B$. The '$\cap$' symbol is the symbol for **intersection**.

The event A or B can be written as $A \cup B$. The '$\cup$' symbol is the symbol for **union**.

The event not A can be written as A'. This is also called the **complement** of A.

2 The probability that B occurs given that A has already occurred is written as $P(B|A)$.

For independent events, $P(A|B) = P(A|B') = P(A)$, and $P(B|A) = P(B|A') = P(B)$.

3 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

4 $P(B|A) = \dfrac{P(B \cap A)}{P(A)}$ so $P(B \cap A) = P(B|A) \times P(A)$